MATHEMATICS MAGAZINE

CONTENTS

VOLUME 39 • SEPTEMBER 1966 • NUMBER 4

MM

MATHEMATICS MAGAZINE

ROY DUBISCH, *EDITOR*

EDITORIAL CORRESPONDENCE should be sent to the Editor, ROY DUBISCH, Department of Mathematics, University of Washington, Seattle, Washington 98105. Articles should be typewritten and double-spaced on $8\frac{1}{2}$ by 11 paper. The greatest possible care should be taken in preparing the manuscript, and authors should keep a complete copy. Figures should be drawn on separate sheets in India ink and of a suitable size for photographing.

NOTICE OF CHANGE OF ADDRESS and other subscription correspondence should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214.

ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214.

The MATHEMATICS MAGAZINE is published by the Mathematical Association of America at Buffalo, New York, bi-monthly except July-August. Ordinary subscriptions are: 1 year \$3.00; 2 years \$5.75; 3 years \$8.50; 5 years \$13.00. Members of the Mathematical Association of America and of Mu Alpha Theta may subscribe at the special rate of 2 years for \$5.00. Single copies are 65¢.

Second class postage paid at Buffalo, New York and additional mailing offices.

THE CIRCULAR FUNCTION(S)

W. F. EBERLEIN, University of Rochester

1. Introduction. The proper treatment of the circular functions in a calculus course remains a problem. No matter how pedagogically justified an intuitive first approach may be, there is an obligation to put matters eventually on the sound arithmetic basis necessary for real and complex variable theory. The classical geometric procedure translates so clumsily into arithmetic language, however, that the obligation, when recognized, is usually met by *ad hoc* definition of the sine and cosine as power series. Contact with geometry is then lost. $\pi/2$ is defined as the smallest positive zero of the cosine, and the periodicity becomes an analytic *tour de force.*

We outline first a modern geometric approach that assumes the minimum background in trigonometry, yet penetrates to the roots of the problem. Then we show how to invert and translate into simple analytic language. The only price we pay is an obvious and easy extension of elementary calculus to complex-valued functions of a *real* variable. The time is long overdue for catching up with the engineers and physicists in this respect.

2. Geometric approach. Consider the following curve in the complex plane C:

$$z = x + iy = \frac{1 + it}{1 - it} = -1 + \frac{2}{1 - it} \quad (-\infty < t < \infty). \tag{1}$$

Then $z\bar{z} = (1+t^2)/(1+t^2) = 1$, and the curve lies on the unit circle. Clearly the point $(-1, 0)$ is never reached, but all other points on $|z| = 1$ are. For if $|z| = 1$ but $z \neq -1$ one can solve (1) for t to obtain

$$t = \frac{z - 1}{i(z + 1)} = \frac{(z - 1)(\bar{z} + 1)}{i(z + 1)(\bar{z} + 1)} = \frac{z - \bar{z}}{i(2 + z + \bar{z})} = \frac{y}{x + 1}, \tag{2}$$

a real number. Thus (1) defines a one-to-one continuous mapping of the line onto the unit circle minus the point $(-1, 0)$. (Note that the rational points on the circle other than $(-1, 0)$ correspond to rational values of t.) Moreover, since (2) implies that the line $Y = t(X+1)$ passes through the points $(-1, 0)$, $(0, t)$, and (x, y), the parameter t admits the simple geometric interpretation shown in Fig. 1.

Let $\theta = s$ be the signed radian measure of $\angle AOP$. Then it follows from the figure that

$$x = \cos\theta, \qquad y = \sin\theta \tag{3}$$

$$t = \tan\frac{\theta}{2}; \tag{4}$$

while separating real and imaginary parts in (1) yields

$$x = \frac{1 - t^2}{1 + t^2} \quad \text{and} \quad y = \frac{2t}{1 + t^2}. \tag{5}$$

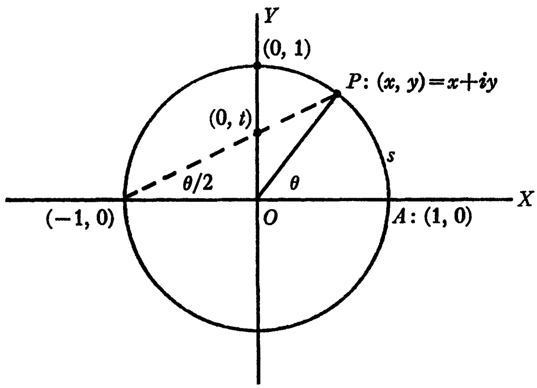


FIG. 1.

(Equations (3), (4), (5) are, of course, the familiar half-angle substitutions introduced by Weierstrass to integrate rational functions of sine, cosine.)

To express θ in terms of t take differentials in (1), conjugate, and multiply;

$$dx + idy = \frac{2i}{(1 - it)^2}\,dt,$$

$$dx - idy = \frac{-2i}{(1 + it)^2}\,dt,$$

$$d\theta^2 = ds^2 = dx^2 + dy^2 = \frac{4dt^2}{(1 + t^2)^2}\,.$$

Since θ is an increasing function of t, it follows that

(6) $$d\theta = \frac{2dt}{1 + t^2}\,.$$

Because $\theta = 0$ when $t = 0$, we have

(7) $$\theta = 2\int_0^t \frac{du}{1 + u^2}\,.$$

Now it follows from the figure that $\theta = \pi/2$ when $t = 1$ and $\theta \uparrow \pi$ when $t \uparrow \infty$; whence

(8a) $$\frac{\pi}{4} = \int_0^1 \frac{du}{1 + u^2}$$

(8b) $$\frac{\pi}{2} = \int_0^\infty \frac{du}{1 + u^2}\,.$$

3. The analytic theory. We have made three major assumptions above: (A) an arc of a circle has a length; (B) an arc of arbitrary length can be laid off on the circle; (C) arc length is given by the usual formula. Although (A) and (C) are demonstrated in most better textbooks, (B) is usually overlooked—even by so acute a mathematician as Hardy [**6**]. We now abandon the suggestive world of geometry for that of pure analysis to develop a rigorous theory of circular functions, yet essentially obtain (B) in passing.

DEFINITION 1.

(8a)
$$\frac{\pi}{4} = \int_0^1 \frac{du}{1+u^2}.$$

The change of variable $u=v^{-1}$ then yields

$$\int_1^\infty \frac{du}{1+u^2} \equiv \lim_{t\to\infty} \int_1^t \frac{du}{1+u^2} = \lim_{t\to\infty} \int_{t^{-1}}^1 \frac{dv}{1+v^2} = \int_0^1 \frac{dv}{1+v^2} = \frac{\pi}{4}$$

whence

(8b)
$$\frac{\pi}{2} = \int_0^\infty \frac{du}{1+u^2}.$$

(For the role of this formula in harmonic analysis see [**5**].)

Consider the function

DEFINITION 2.

(7)
$$\theta = \theta(t) = 2\int_0^t \frac{du}{1+u^2} (-\infty < t < \infty).$$

Then θ is an odd increasing function of t with positive derivative

(6)
$$\frac{d\theta}{dt} = \frac{2}{1+t^2}$$

and $\theta \uparrow \pi$ when $t \uparrow \infty$. Hence we can invert to obtain $t=t(\theta)$ as an odd increasing differentiable function of $\theta(-\pi<\theta<\pi)$. Set

DEFINITION 3.

(1)
$$\operatorname{cis}\theta = \frac{1+it}{1-it} = -1 + \frac{2}{1-it} (-\pi < \theta < \pi).$$

Then $|\operatorname{cis}\theta| = 1$ and in particular

(9)
$$\operatorname{cis} 0 = 1;$$

cis becomes a continuous function on the closed interval $[-\pi, \pi]$ on setting

$$\operatorname{cis}(\pm\pi) \equiv \lim_{\theta\to\pm\pi} \operatorname{cis}\theta = \lim_{t\to\pm\infty} \frac{1+it}{1-it} = -1.$$

We can now extend the definition of cis θ to arbitrary real values of θ by the obvious requirement

DEFINITION 4. $\text{cis}(\theta+2\pi)=\text{cis}\,\theta$.

In the open interval $-\pi<\theta<\pi$

$$\text{cis}'\,\theta = \frac{(d\,\text{cis}\,\theta)/dt}{d\theta/dt} = \frac{2i/(1-it)^2}{2/(1+t^2)} = i\frac{1+it}{1-it}$$

$$\text{cis}'\,\theta = i\,\text{cis}\,\theta. \tag{10}$$

This formula trivially stays valid at all θ other than odd multiples of π. The validity at these remaining points results from a familiar corollary of the mean value theorem: *Suppose f is continuous in some neighborhood of a and differentiable in some deleted neighborhood of a. If $\lim_{x\to a} f'(x)=A$ exists, then $f'(a)$ exists and equals A.*

The functions $f=a$ cis ($a\in C$) clearly satisfy the differential equation

$$f' = if \qquad (\text{differentiable } f\colon R\to C). \tag{*}$$

But these are the only solutions of (*):

THEOREM. *If f satisfies (*), then $f(\theta)=f(0)$ cis θ. In particular, $f(0)=1$, implies $f=$cis.*

Proof.

$$D_\theta \frac{f(\theta)}{\text{cis}\,\theta} = \frac{i\,\text{cis}\,\theta f(\theta) - if(\theta)\,\text{cis}\,\theta}{(\text{cis}\,\theta)^2} = 0.$$

Hence $f(\theta)/\text{cis}\,\theta \equiv f(0)/\text{cis}\,0 = f(0)$ or $f(\theta)=f(0)\,\text{cis}\,\theta$.

Set $f(\theta)=\text{cis}(\alpha+\theta)$. Then $f'=if$ and $f(0)=\text{cis}\,\alpha$, whence

$$\text{cis}(\alpha+\theta) = \text{cis}\,\alpha\cdot\text{cis}\,\theta. \tag{11}$$

In particular, $\text{cis}\,(-\theta)\,\text{cis}\,\theta=\text{cis}\,0=1$ or $\text{cis}\,(-\theta)=(\text{cis}\,\theta)^{-1}=\overline{\text{cis}\,\theta}$ (since $|\text{cis}\,\theta|=1$)—a result we need later.

Now let $\cos\theta$ and $\sin\theta$ be the real and imaginary parts respectively of cis θ:

DEFINITION 5.

$$\text{cis}\,\theta = \cos\theta + i\sin\theta. \tag{3}$$

Separating real and imaginary parts in (10) then yields the differentiation formulae

$$\cos'\theta = -\sin\theta, \qquad \sin'\theta = \cos\theta; \tag{12}$$

and in (11), the addition formulae

$$\begin{aligned} \cos(\alpha+\theta) &= \cos\alpha\cos\theta - \sin\alpha\sin\theta \\ \sin(\alpha+\theta) &= \sin\alpha\cos\theta + \cos\alpha\sin\theta. \end{aligned} \tag{13}$$

Now adopt the more picturesque notation

DEFINITION 6. $e^{i\theta}=\text{cis }\theta=\cos\theta+i\sin\theta\ (-\infty<\theta<\infty)$.

Equations (9), (10), (11) then take the transparent form

$$e^{i0} = 1 \tag{9'}$$

$$D_\theta e^{i\theta} = ie^{i\theta} \tag{10'}$$

$$e^{i(\alpha+\theta)} = e^{i\alpha}e^{i\theta}. \tag{11'}$$

Moreover, our theorem implies that the only definition of $e^{i\theta}$ (as a complex-valued function of the real variable θ) satisfying (9′) and (10′) is the one we have made: $e^{i\theta}=\text{cis }\theta$.

Since $e^{-i\theta}\equiv\text{cis }(-\theta)=(\text{cis }\theta)^{-1}=\overline{\text{cis }\theta}=\cos\theta-i\sin\theta$, we have

$$\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}) \quad\text{and}\quad \sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}). \tag{14}$$

Finally solve the equation

$$e^{i\theta} = \frac{1+it}{1-it}\ (-\pi<\theta<\pi)$$

for t to obtain

$$t = \frac{e^{i\theta}-1}{i(e^{i\theta}+1)} = \frac{(e^{i\theta/2}-e^{-i\theta/2})/2i}{(e^{i\theta/2}+e^{-i\theta/2})/2} = \frac{\sin\theta/2}{\cos\theta/2} \equiv \tan\theta/2. \tag{4}$$

Note that we have established the mapping $\theta\to e^{i\theta}$ as a one-to-one mapping of the open interval $(-\pi, \pi)$ onto the unit circle minus the point $(-1, 0)$, the foundation for polar coordinates.

Work sponsored by National Science Foundation Grant N.S.F. GP-4021.

References

1. R. P. Agnew, Differential equations, 2nd ed., McGraw-Hill, New York, 1960.

2. L. V. Ahlfors, Complex analysis, McGraw-Hill, New York, 1953.

3. N. Bourbaki, Fonctions d'une Variable Réelle, vol. 9, Chapitre 3, (Actualités Scientifiques et Industrielles No. 1074), Paris, 1949.

4. C. Caratheodory, Theory of functions, vol. 1, Chelsea, New York, 1958.

5. W. F. Eberlein, The elementary transcendental functions, Amer. Math. Monthly, 61 (1954) 386–392.

6. G. H. Hardy, Pure mathematics, 10th ed., Cambridge, New York, 1959.

7. F. Klein, Elementary mathematics from an advanced standpoint, vol. 1, Dover, New York, 1961.

8. E. T. Whittaker and G. N. Watson, Modern Analysis (appendix), 4th ed., Cambridge, New York, 1958.

A CONNECTION BETWEEN TWO THEOREMS IN THE THEORY OF RIEMANN-STIELTJES INTEGRATION

G. T. CARGO, Syracuse University

Because of its connection with the Euler-Maclaurin sum formula, the following theorem is included in most developments of Riemann-Stieltjes integration. (Cf. [1], pp. 200–202, and [2], p. 256.)

THEOREM A. *Let $a, c_0, c_1, \cdots, c_n, b$ be real numbers such that $a=c_0<c_1<\cdots<c_n=b$; let g be a complex-valued function on the closed interval $[a, b]$ that is constant on each open interval $]c_{k-1}, c_k[$ $(k=1, 2, \cdots, n)$; and let f be a bounded complex-valued function on $[a, b]$ that is continuous relative to $[a, b]$ at each discontinuity of g in $[a, b]$. Then f is integrable with respect to g on $[a, b]$, and*

$$\int_a^b fdg = f(a)\{g(a+) - g(a)\} + \sum_{k=1}^{n-1} f(c_k)\{g(c_k+) - g(c_k-)\} + f(b)\{g(b) - g(b-)\}. \tag{1}$$

The purpose of this note is to exhibit an interrelationship between Theorem A and the following theorem, which states that a Riemann-Stieltjes integral is unaltered if the values of the integrand are altered at a finite number of points where the integrator is continuous. (Cf. [3], p. 342.)

THEOREM B. *Let f and g be bounded complex-valued functions on $[a, b]$ such that f is integrable with respect to g on $[a, b]$, and let f^* be a complex-valued function on $[a, b]$ such that $f^*(x)=f(x)$ for all x in $[a, b]$ except for a finite number of points x in $[a, b]$ where g is continuous relative to $[a, b]$. Then f^* is integrable with respect to g on $[a, b]$, and*

$$\int_a^b f^*dg = \int_a^b fdg.$$

First, let us show that Theorem A implies Theorem B. Setting $h=f^*-f$, we note that it is sufficient to prove that

$$\int_a^b hdg = 0. \tag{2}$$

From Theorem A, we conclude that

$$\int_a^b gdh = g(a)\{0 - h(a)\} + 0 + g(b)\{h(b) - 0\}. \tag{3}$$

The theorem on integration by parts shows that h is integrable with respect to g and that

$$\int_a^b hdg = h(b)g(b) - h(a)g(a) - \int_a^b gdh. \tag{4}$$

Combining (3) and (4), we obtain (2), as desired.

To infer Theorem A from Theorem B, let G_k denote the constant value of g

on $]c_{k-1}, c_k[$ $(k=1, 2, \cdots, n)$, use integration by parts and Theorem B to deduce that

$$(5)\quad \begin{aligned}\int_{c_{k-1}}^{c_k} f dg &= f(c_k)g(c_k) - f(c_{k-1})g(c_{k-1}) - G_k\{f(c_k) - f(c_{k-1})\}\\ &= f(c_k)g(c_k) - f(c_{k-1})g(c_{k-1}) - f(c_k)g(c_k-) + f(c_{k-1})g(c_{k-1}+),\\ &\qquad (k = 1, 2, \cdots, n)\end{aligned}$$

and then manipulate equations (5) in an obvious way to obtain (1).

This work was supported by the National Science Foundation through Grant GP 1086.

References

1. T. M. Apostol, Mathematical Analysis, Addison-Wesley, Reading, Mass., 1957.
2. H. Kestelman, Modern Theories of Integration, 2nd rev. ed., Dover, New York, 1960.
3. R. A. Rankin, An Introduction to Mathematical Analysis, Macmillan, New York, 1963.

ELLIPSE OF LEAST ECCENTRICITY

N. X. VINH, University of Colorado

This note presents a problem of geometry with application to astronautics. In Figure 1, O is the center of the earth and P is a point on the top of the sensible atmosphere at a distance R from O. A satellite, following an elliptic trajectory, re-enters the atmosphere at the point P and its velocity $\vec{V}$ makes an angle γ with the local horizontal which is the perpendicular to OP at the point P. Let F be the second focus of the elliptic trajectory, the first focus being the point O by Kepler's first law. It is known from the geometry of the conic that $\measuredangle OPF=2\gamma$. Hence, for a given angle γ (re-entry angle), the locus of F is the ray $P\Delta$ making an angle 2γ with the direction PO. Of the infinite number of possible trajectories one looks for the elliptic flight path with the least numerical eccentricity.

Since P is a point on the ellipse, $OP+PF=2a=$major axis, and $OF=2c$ $=$focal distance. Therefore

$$e = \frac{c}{a} = \frac{OF}{OP + PF}.$$

The problem is to find the point F on the straight line $P\Delta$ such that the ratio $OF/(OP+PF)$ is a minimum.

(1) *Trigonometric solution*: Let $x=\measuredangle POF$. By the law of sines

$$\frac{OF}{\sin 2\gamma} = \frac{PF}{\sin x} = \frac{R}{\sin (2\gamma + x)}.$$

on $]c_{k-1}, c_k[$ $(k=1, 2, \cdots, n)$, use integration by parts and Theorem B to deduce that

$$\int_{c_{k-1}}^{c_k} f dg = f(c_k)g(c_k) - f(c_{k-1})g(c_{k-1}) - G_k\{f(c_k) - f(c_{k-1})\}$$

$$(5) \qquad = f(c_k)g(c_k) - f(c_{k-1})g(c_{k-1}) - f(c_k)g(c_k-) + f(c_{k-1})g(c_{k-1}+),$$

$$(k = 1, 2, \cdots, n)$$

and then manipulate equations (5) in an obvious way to obtain (1).

This work was supported by the National Science Foundation through Grant GP 1086.

References

1. T. M. Apostol, Mathematical Analysis, Addison-Wesley, Reading, Mass., 1957.
2. H. Kestelman, Modern Theories of Integration, 2nd rev. ed., Dover, New York, 1960.
3. R. A. Rankin, An Introduction to Mathematical Analysis, Macmillan, New York, 1963.

ELLIPSE OF LEAST ECCENTRICITY

N. X. VINH, University of Colorado

This note presents a problem of geometry with application to astronautics. In Figure 1, O is the center of the earth and P is a point on the top of the sensible atmosphere at a distance R from O. A satellite, following an elliptic trajectory, re-enters the atmosphere at the point P and its velocity $\vec{V}$ makes an angle γ with the local horizontal which is the perpendicular to OP at the point P. Let F be the second focus of the elliptic trajectory, the first focus being the point O by Kepler's first law. It is known from the geometry of the conic that $\measuredangle OPF=2\gamma$. Hence, for a given angle γ (re-entry angle), the locus of F is the ray $P\Delta$ making an angle 2γ with the direction PO. Of the infinite number of possible trajectories one looks for the elliptic flight path with the least numerical eccentricity.

Since P is a point on the ellipse, $OP+PF=2a=$ major axis, and $OF=2c$ $=$ focal distance. Therefore

$$e = \frac{c}{a} = \frac{OF}{OP + PF}.$$

The problem is to find the point F on the straight line $P\Delta$ such that the ratio $OF/(OP+PF)$ is a minimum.

(1) *Trigonometric solution*: Let $x = \measuredangle POF$. By the law of sines

$$\frac{OF}{\sin 2\gamma} = \frac{PF}{\sin x} = \frac{R}{\sin (2\gamma + x)}.$$

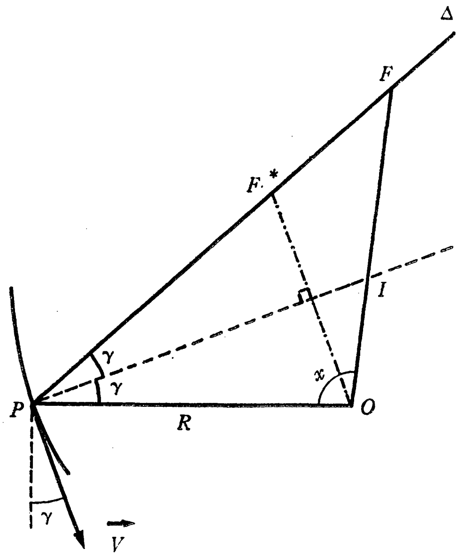


FIG. 1.

Therefore

$$OF = \frac{R \sin 2\gamma}{\sin (2\gamma + x)}, \qquad PF = \frac{R \sin x}{\sin (2\gamma + x)}$$

and

$$e = \frac{\sin 2\gamma}{\sin x + \sin (2\gamma + x)}$$

$$= \frac{\sin 2\gamma}{\sin 2\gamma \cos x + (1 + \cos 2\gamma) \sin x}$$

$$= \frac{\sin \gamma}{\sin \gamma \cos x + \sin x \cos \gamma} = \frac{\sin \gamma}{\sin (\gamma + x)} .$$

Hence e is a minimum when $x = \pi/2 - \gamma$. The point F is at F^*.

(2) *Geometric solution.* Let I be the intersection of OF and the bisector of the angle OPF. Then

$$\frac{IF}{IO} = \frac{PF}{PO}$$

or

$$\frac{IF + IO}{IO} = \frac{PF + PO}{PO} \quad \text{and} \quad \frac{OF}{OI} = \frac{OP + PF}{OP} .$$

Therefore

$$e = \frac{OF}{OP + PF} = \frac{OI}{OP}.$$

Since OP is fixed, the ratio is a minimum when OI is a minimum, i.e., when OF is perpendicular to the bisector of the angle OPF.

This result is from a study of Orbital Trajectories supported by Advanced Research Projects Agency under contract DA 31 124 ARO (D) 139.

TWO GROUPS ASSOCIATED WITH RULED SURFACES

FRANK H. SWEET, Waterloo University College

1. Introduction. The hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{1}$$

and the hyperbolic paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z \tag{2}$$

are ruled surfaces i.e., through any point on the particular surfaces two lines lying entirely in that surface may be drawn. These lines are termed the rulings. String models of the above surfaces may then be made by joining pairs of points lying on each ruling, for a sufficient number of these rulings. A particular method for solving for these points leads to some interesting results.

2. Hyperboloid of one sheet. We write (1) in the form

$$\left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right)\left(1 - \frac{y}{b}\right); \tag{3}$$

then we may "split" (3) as follows

$$\left.\begin{cases} \dfrac{x}{a} + \dfrac{z}{c} = k\left(1 + \dfrac{y}{b}\right) \\ k\left(\dfrac{x}{a} - \dfrac{z}{c}\right) = 1 - \dfrac{y}{b} \end{cases}\right\} R_1 \qquad \left.\begin{cases} \dfrac{x}{a} + \dfrac{z}{c} = m\left(1 - \dfrac{y}{b}\right) \\ m\left(\dfrac{x}{a} - \dfrac{z}{c}\right) = 1 + \dfrac{y}{b} \end{cases}\right\} R_2$$

where k and m are scalars. Each equation in R_1 represents a plane and hence, as a pair, they determine a line—ruling one (R_1). Similarly the pair of equations in R_2 determine ruling two.

Therefore

$$e = \frac{OF}{OP + PF} = \frac{OI}{OP}.$$

Since OP is fixed, the ratio is a minimum when OI is a minimum, i.e., when OF is perpendicular to the bisector of the angle OPF.

This result is from a study of Orbital Trajectories supported by Advanced Research Projects Agency under contract DA 31 124 ARO (D) 139.

TWO GROUPS ASSOCIATED WITH RULED SURFACES

FRANK H. SWEET, Waterloo University College

1. Introduction. The hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{1}$$

and the hyperbolic paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z \tag{2}$$

are ruled surfaces i.e., through any point on the particular surfaces two lines lying entirely in that surface may be drawn. These lines are termed the rulings. String models of the above surfaces may then be made by joining pairs of points lying on each ruling, for a sufficient number of these rulings. A particular method for solving for these points leads to some interesting results.

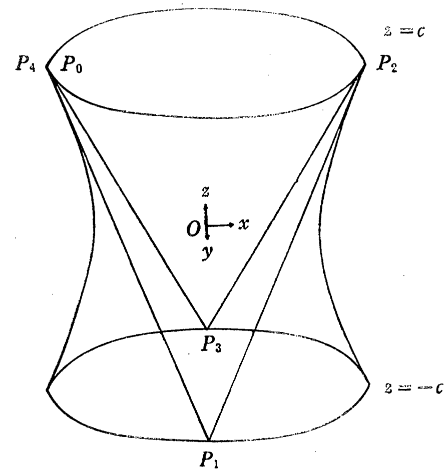
2. Hyperboloid of one sheet. We write (1) in the form

$$\left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right)\left(1 - \frac{y}{b}\right); \tag{3}$$

then we may "split" (3) as follows

$$\left.\begin{cases} \dfrac{x}{a} + \dfrac{z}{c} = k\left(1 + \dfrac{y}{b}\right) \\ k\left(\dfrac{x}{a} - \dfrac{z}{c}\right) = 1 - \dfrac{y}{b} \end{cases}\right\} R_1 \qquad \left.\begin{cases} \dfrac{x}{a} + \dfrac{z}{c} = m\left(1 - \dfrac{y}{b}\right) \\ m\left(\dfrac{x}{a} - \dfrac{z}{c}\right) = 1 + \dfrac{y}{b} \end{cases}\right\} R_2$$

where k and m are scalars. Each equation in R_1 represents a plane and hence, as a pair, they determine a line—ruling one (R_1). Similarly the pair of equations in R_2 determine ruling two.



We start with a point (x_0, y_0, z_0) such that

$$\frac{x_0}{a} = \alpha_0, \qquad \frac{y_0}{b} = \beta_0, \qquad \frac{z_0}{c} = \gamma_0 = 1.$$

(In the following $x_i/a=\alpha_i$, $y_i/b=\beta_i$, $z_i/c=\gamma_i$).

The ruling R_1 through this point is determined and then the point where R_1 pierces the plane $z=-c$ is found. We then take this latter point and determine the ruling R_2 through it. This passes through the plane $z=c$ at some third point, through which a new ruling R_1 passes, etc. For $x_0/a=\alpha_0$, $y_0/b=\beta_0$, $z_0/c=\gamma_0=1$, R_1 gives $\alpha_0+1=k(1+\beta_0)$ and $k(\alpha_0-1)=1-\beta_0$ whence

$$(4) \qquad \alpha_0 = \frac{k^2+2k-1}{k^2+1}, \qquad \beta_0 = \frac{-k^2+2k+1}{k^2+1}, \quad \text{and} \quad \gamma_0 = 1.$$

Assuming k to be determined for $\alpha_0, \beta_0, \gamma_0$, it follows that R_1 passes through the plane $z=-c$ at the "point" $(\alpha_1, \beta_1, \gamma_1)$ determined by setting $z=-c$ in R_1:

$$(5) \qquad \alpha_1 - 1 = k(1+\beta_1), \qquad k(\alpha_1+1) = 1-\beta_1$$

whence

$$(6) \qquad \alpha_1 = \frac{-k^2+2k+1}{k^2+1}, \qquad \beta_1 = \frac{-k^2-2k+1}{k^2+1}.$$

A comparison of (6) and (4) gives $\alpha_1=\beta_0$ and $\beta_1=-\alpha_0$. In addition $\gamma_1=-1$. If the values α_1, β_1, and γ_1 are inserted in R_2, then m is determined and we obtain

$$(7) \qquad \alpha_1 = \frac{-m^2+2m+1}{m^2+1}, \qquad \beta_1 = \frac{m^2+2m-1}{m^2+1}.$$

Setting $z=c$ in R_2, to determine where the ruling pierces the plane, yields

$$\alpha_2 = \frac{m^2+2m-1}{m^2+1} = \beta_1, \qquad \beta_2 = \frac{m^2-2m-1}{m^2+1} = -\alpha_1 \tag{8}$$

upon comparison with (7).

We need proceed no further. The relationships $(\alpha_1, \beta_1, -1) = (\beta_0, -\alpha_0, -(1))$ and $(\alpha_2, \beta_2, 1) = (\beta_1, -\alpha_1, -(-1))$ suggest the existence of a matrix operator T such that $TP_n = P_{n+1}$ where

$$T = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$P_n = \begin{pmatrix} \alpha_n \\ \beta_n \\ (-1)^n \end{pmatrix}.$$

By proceeding as above, it is readily verified that P_3 is a solution to the problem and that $P_4 = P_0$. This also follows from the fact that $T^4 = I$ where I is the 3×3 identity matrix. Hence, the operator T generates a cyclic group of order four.

Finally the sequence of "points" to be joined to form the model is (dropping the subscript 0): $(\alpha, \beta, 1)$, $(\beta, -\alpha, -1)$, $(-\alpha, -\beta, 1)$, $(-\beta, \alpha, -1)$, $(\alpha, \beta, 1)$, where $\alpha^2+\beta^2=2$ since $\gamma^2=1$.

3. The hyperbolic paraboloid. In this case we take R_1 and R_2 to be

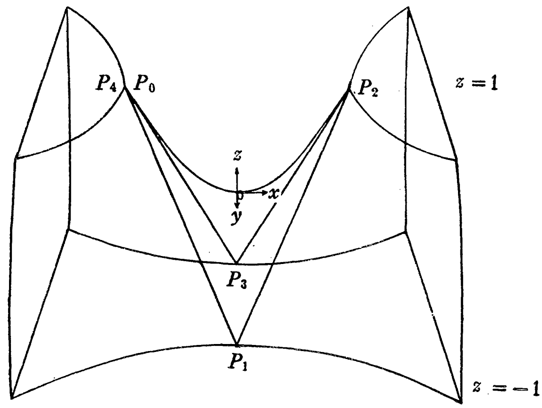
$$\left.\begin{cases} \dfrac{x}{a}+\dfrac{y}{b} = kz \\ k\left(\dfrac{x}{a}-\dfrac{y}{b}\right) = 1 \end{cases}\right\} R_1 \qquad \left.\begin{cases} m\left(\dfrac{x}{a}+\dfrac{y}{b}\right) = 1 \\ \dfrac{x}{a}-\dfrac{y}{b} = mz \end{cases}\right\} R_2$$

and work on the planes $z = \pm 1$. Using the same notation and technique as above, we derive the sequence

$$(\alpha, \beta, 1), (-\beta, -\alpha, -1), (-\alpha, -\beta, 1), (\beta, \alpha, -1), (\alpha, \beta, 1), \text{ where } \alpha^2 - \beta^2 = 1.$$

In contrast to the case of the hyperboloid of one sheet, *two* matrix operators are needed here, and are defined as follows:

$$S = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$



Then

$$
\begin{aligned}
P_1 &= S\ P_0 \\
P_2 &= T S\ P_0 \\
P_3 &= S\ T S\ P_0 \qquad (9) \\
P_4 &= T S\ T S\ P_0 \quad \text{etc.} \\
&= P_0.
\end{aligned}
$$

This set of transformations is isomorphic to Klein's *Vierergruppe.*

CONVEX SOLUTIONS OF IMPLICIT RELATIONS

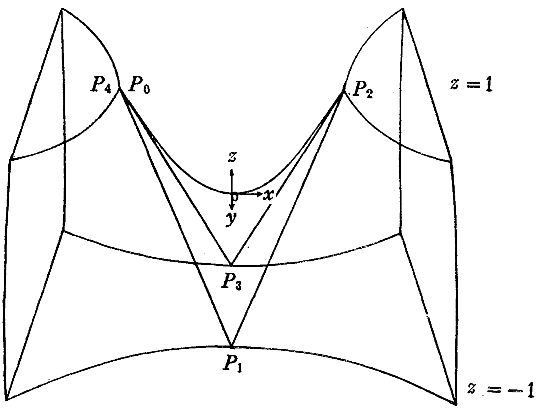
WILLIAM A. BROCK AND RUSSELL G. THOMPSON, University of Missouri

Many times in applied work an analyst wants to know when a solution $x_i = g_i(y_1, \cdots, y_n)$ $(i = 1, 2, \cdots, m)$ exists which satisfies the implicit (vector valued) relation $F(x, y) = 0$; and furthermore what properties this solution possesses. The conditions needed on $F(x, y) = 0$ to insure uniqueness, continuity, and differentiability are well known; but those conditions needed to guarantee a convex solution are not known.

It is our purpose to derive in this article a set of necessary and sufficient conditions which will insure the convexity of the solution of $F(x, y) = 0$.

We first dispose of some definitions.

DEFINITION 1. $E_n \equiv n$ *dimensional Euclidean space.*



Then

$$
\begin{aligned}
P_1 &= S\ P_0 \\
P_2 &= T\,S\ P_0 \\
P_3 &= S\ T\,S\ P_0 \\
P_4 &= T\,S\ T\,S\ P_0 \quad \text{etc.} \\
&= P_0.
\end{aligned}
\tag{9}
$$

This set of transformations is isomorphic to Klein's *Vierergruppe.*

CONVEX SOLUTIONS OF IMPLICIT RELATIONS

WILLIAM A. BROCK AND RUSSELL G. THOMPSON, University of Missouri

Many times in applied work an analyst wants to know when a solution $x_i = g_i(y_1, \cdots, y_n)$ $(i=1, 2, \cdots, m)$ exists which satisfies the implicit (vector valued) relation $F(x, y) = 0$; and furthermore what properties this solution possesses. The conditions needed on $F(x, y) = 0$ to insure uniqueness, continuity, and differentiability are well known; but those conditions needed to guarantee a convex solution are not known.

It is our purpose to derive in this article a set of necessary and sufficient conditions which will insure the convexity of the solution of $F(x, y) = 0$.

We first dispose of some definitions.

DEFINITION 1. *$E_n \equiv n$ dimensional Euclidean space.*

DEFINITION 2. *The vector valued function* $\vec{f}=(f_1, f_2, \cdots, f_n)$ *of a real variable* $x=(x_1, \cdots, x_m)$ *is said to be convex if each component* f_j *is convex,* i.e., $f_j(\alpha x_1 + (1-\alpha)x_2) \leqq \alpha f_j(x_1) + (1-\alpha)f_j(x_2) \forall j$, $\alpha \in [0, 1]$, x_1, x_2 *in the domain of* f_j. *The function f is said to be concave if its negative is convex.*

DEFINITION 3. $J_f(x) \equiv$ *the Jacobian of f evaluated at the point x,* i.e., $J_f(x) = \det |\partial f_i/\partial x_j|$ $(i=1, 2, \cdots, n; j=1, 2, \cdots, n)$.

DEFINITION 4. *A set* $\overline{X}$ *is said to be convex if* $x_1, x_2 \in \overline{X}$ *imply* $\alpha x_1 + \beta x_2 \in \overline{X}$, $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha + \beta = 1$.

We will now state some well-known theorems.

THE IMPLICIT FUNCTION THEOREM. [1, p. 147]. *Let* $\vec{f} \equiv (f_1, \cdots, f_n)$ *be defined on an open set* $S \subseteq E_{n+k}$ *where each* $f_j = f_j(x_1, \cdots, x_n; t_1, \cdots, t_k)$ *with range* $R \subseteq E_n$. *Suppose f possesses continuous first partial derivatives on S. Let* (x_0, t_0) *be an element of S such that* $f(x_0, t_0) = 0$ *and* $J_f(x_0) \neq 0$. *Then there exists an open set* $T \subseteq E_k$ *where there exists a vector valued function* $\vec{g} = (g_1, \cdots, g_n)$ *satisfying the following conditions:*

(1) $\vec{g}$ *has domain T and range* $G \subseteq E_n$
(2) $\vec{g}$ *possesses continuous first partial derivatives in T*
(3) $\vec{g}(t_0) = x_0$ *and*
(4) $\vec{f}(\vec{g}(t), t) = 0$ *for all* $t \in T$.

THE CONVEXITY THEOREM. [2, p. 406]. *Let the function* $f(x) = f(x_1, \cdots, x_n)$ *be defined on a convex set* $\overline{X} \subseteq E_n$ *and let f possess second partial derivatives in* $\overline{X}$. *Then f is convex in* $\overline{X}$ *if and only if the matrix* $Q(x) = \|\partial^2 f/\partial x_i \partial x_j\|$ *is positive semi-definite for every* $x \in \overline{X}$.

These two theorems shall provide the foundation upon which we shall base our analysis. Let's consider two simple cases of implicit relations. They basically can be used as a guide to our analysis.

Special cases for $F(x, t)$. We shall first consider the case where x, t are both one dimensional.

Claim I. Suppose the hypotheses of the implicit function theorem are satisfied. Further suppose T is a convex set. Then the "solution" $x = x(t)$ of the relation $F(x, t) = 0$ is convex in T if and only if the quantity

$$Q = \frac{\partial F}{\partial t}\frac{\partial^2 F}{\partial x \partial t} - \frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial t^2} \geqq 0. \tag{5}$$

Proof. It is well known that

$$\frac{\partial x}{\partial t} = -\frac{\dfrac{\partial F}{\partial t}}{\dfrac{\partial F}{\partial x}} \quad \text{and} \quad \frac{\partial^2 x}{\partial t^2} = -\left[\frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial t^2} - \frac{\partial F}{\partial t}\frac{\partial^2 F}{\partial x \partial t}\right] \Big/ \left(\frac{\partial F}{\partial x}\right)^2 \tag{6}$$

for all $t \in T$.

Hence

$$\frac{\partial^2 x}{\partial t^2} = \left[\frac{\partial F}{\partial t}\frac{\partial^2 F}{\partial x \partial t} - \frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial t^2}\right] \Big/ (\partial F/\partial x)^2 \tag{7}$$

But by the convexity theorem, $x(t)$ is convex in the convex set T if and only if $\partial^2 x/\partial t^2 \geqq 0$, i.e., if and only if

$$\frac{\partial F}{\partial t}\frac{\partial^2 F}{\partial x \partial t} - \frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial t^2} \geqq 0. \tag{8}$$

For our second case, we will consider the case where t is k dimensional and x is one dimensional.

Claim II. Suppose the hypotheses of the implicit function theorem and the convexity theorem are satisfied. Further suppose T is convex. Then the solution $x = x(t)$ to the implicit relation $F(x, t) = 0$ is convex if and only if the $k \times k$ matrix

$$Q = \left\| \frac{\partial F}{\partial t_i}\frac{\partial^2 F}{\partial x \partial t_j} - \frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial t_i \partial t_j} \right\| \text{ is positive semi-definite } \forall t \in T. \tag{9}$$

Proof. It is well known that

$$\frac{\partial x}{\partial t_i} = -\frac{\dfrac{\partial F}{\partial t_i}}{\dfrac{\partial F}{\partial_x}}, \qquad (i = 1, 2, \cdots, k). \tag{10}$$

Hence for each $ij = 1, 2, \cdots, k$

$$\frac{\partial^2 x}{\partial t_j \partial t_i} = \left[\frac{\partial F}{\partial t_i}\frac{\partial^2 F}{\partial x \partial t_j} - \frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial t_i \partial t_j}\right] \Big/ (\partial F/\partial x)^2. \tag{11}$$

The matrix $\|\partial^2 x/\partial t_i \partial t_j\|$ must be positive semi-definite for all $t \in T$ for the convexity of $x(t)$ in T. Hence,

$$Q = \left\| \frac{\partial F}{\partial t_i}\frac{\partial^2 F}{\partial x \partial t_j} - \frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial t_i \partial t_j} \right\| \tag{12}$$

must be positive semi-definite for all $t \in T$.

We will consider next the general case where x is n dimensional and t is k dimensional. The above notation is quite cumbersome though; so from this point forward we shall use the following notation for partial derivatives:

$$F_x = \frac{\partial F}{\partial x}, \quad F_{t_i t_j} = \frac{\partial^2 F}{\partial t_i \partial t_j}, \quad F_{t_i x} = \frac{\partial^2 F}{\partial t_i \partial x},$$

$$F_{ij} = \frac{\partial F_i}{\partial x_j}, \quad X_{it_j} = \frac{\partial x_i}{\partial t_j}, \quad F_{ji} = \frac{\partial F_j}{\partial t_i}.$$

THEOREM 1. *Suppose the hypotheses of the implicit function theorem are satisfied for the implicit relation* $F(x, t) = (F_1(x, t), F_2(x, t), \cdots, F_n(x, t)) = 0$, *where* t *ranges over the convex set* T. *Then the solution* $x = (x_1(t), x_2(t), \cdots, x_n(t))$ *of the relation* $F(x, t) = 0$ *is convex if and only if the matrix* $\|X_{kt_it_j}\| = \| |F| \partial G(k, t_i)/\partial t_j - G(k, t_j)\partial |F| /\partial t_j \|$ *is positive semi-definite* ($k = 1, 2, \cdots, n$). (*The symbol* $G(k, t_j)$ *will be defined below.*)

Proof. Consider $F(x, t) = 0$ for all $t \in T$, i.e.,

$$(13) \qquad \begin{aligned} F_1(x_1, x_2, \cdots, x_n, t_1, t_2, \cdots, t_k) &= 0 \\ F_2(x_1, x_2, \cdots, x_n, t_1, t_2, \cdots, t_k) &= 0 \\ &\vdots \\ F_n(x_1, x_2, \cdots, x_n, t_1, t_2, \cdots, t_k) &= 0. \end{aligned}$$

Differentiating (11) with respect to t_i (fixed i) we have

$$(14) \qquad \begin{bmatrix} F_{11}, F_{12}, \cdots, F_{1n} \\ F_{21}, F_{22}, \cdots, F_{2n} \\ \vdots \\ F_{n1}, F_{n2}, \cdots, F_{nn} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial t_i} \\ \vdots \\ \frac{\partial x_n}{\partial t_i} \end{bmatrix} = - \begin{bmatrix} \frac{\partial F_1}{\partial t_i} \\ \vdots \\ \frac{\partial F_n}{\partial t_i} \end{bmatrix}$$

Using Cramer's Rule, we obtain from (12) for $i = 1$

$$(15) \qquad X_{jt_j} = \frac{1}{|\mathrm{F}|} \begin{vmatrix} F_{11}, \cdots, F_1, j-1, -F_{1t_1}, F_1, j+1, \cdots, F_{1n} \\ \vdots \\ F_{n1}, \cdots, F_n, j-1, -F_{nt_1}, F_n, j+1, \cdots, F_{nn} \end{vmatrix} = \frac{G(j, t_1)}{|F|}, \qquad (j = 1, \cdots, n)$$

where $|F|$ is the determinant of the matrix $\|F_{ij}\|$, and $G(j, t_1)$ represents the determinant in the numerator of (13). For any t_i we obtain a similar expression as (13) viz.

$$(16) \qquad X_{jt_i} = G(j, t_i)/|F|.$$

Now the vector valued function $x = (x_1, x_2, \cdots, x_n)$ is (by definition) convex if and only if each component x_k is convex. Hence each matrix $\|X_{kt_it_j}\|$ must be positive semi-definite. By (13) $X_{kt_i} = G(k, t_i)/|F|$ so

$$(17) \qquad \|X_{kt_it_j}\| = \|[|F| \partial G(k, t_i)/\partial t_j - G(k, t_i)\partial |F| /\partial t_j]/|F|^2\|, \quad (k = 1, 2, \cdots, n).$$

It follows then that the matrix $\|X_{kt_it_j}\|$ is positive semi-definite if and only if the matrix $\| |F| \partial G(k, t_i)/\partial t_j - G(k, t_i)\partial |F| /\partial t_j\|$ is positive semi-definite ($k = 1, 2, \cdots, n$).

References

1. T. M. Apostol, Mathematical Analysis, Addison-Wesley, Reading, Mass., 1957.

2. S. Karlin, Mathematical Methods and Theory in Games, Programming, and Economics, Vol. I, Addison-Wesley, Reading, Mass., 1959.

FINDING THE CUBE ROOT OF BINOMIAL QUADRATIC SURDS

KAIDY TAN, Fukien Normal College, Foochow, China

In a cubic equation $x^3+px+q=0$ the roots may be expressed as $x_1=A+B$, $x_2=A\omega+B\omega^2$, $x_3=A\omega^2+B\omega$, where ω is any one of the imaginary cube roots of unity $(-1\pm i\sqrt{3})/2$, and $A=(-\frac{1}{2}q+\sqrt{D})^{1/3}$, $B=(-\frac{1}{2}q-\sqrt{D})^{1/3}$, $D=q^2/4+p^3/27$. If D is a perfect square then A and B are easily found; otherwise, how do we find the value of A and B? The aim of this paper is to give a streamlined method to solve this problem.

In [2, pp. 70–71] this problem is discussed and the following result is stated: If $(a+\sqrt{b})^{1/3}=x+\sqrt{y}$, then $(a-\sqrt{b})^{1/3}=x-\sqrt{y}$, where a, b, x, y are rational, and $\sqrt{b}$ and $\sqrt{y}$ are irrational. Two illustrative examples are then given. The first example is shown here as Example 1: then I shall give my method to compare with it.

Example 1. Find the cube root of $72-32\sqrt{5}$.

Solution. Assume $(72-32\sqrt{5})^{1/3}=x-\sqrt{y}$; then $(72+32\sqrt{5})^{1/3}=x+\sqrt{y}$. By multiplication, $(5184-1024\times 5)^{1/3}=x^2-y$, that is

$$4 = x^2 - y. \tag{1}$$

Again, $72-32\sqrt{5}=x^3-3x^2\sqrt{y}+3xy-y\sqrt{y}$, whence

$$72 = x^3 + 3xy. \tag{2}$$

From (1) and (2) we obtain $72=x^3+3x(x^2-4)$ so that $x^3-3x=18$. By trial, we find that $x=3$; hence $y=5$, and the cube root is $3-\sqrt{5}$.

I. If $(a\pm\sqrt{b})^{1/3}$ can be expressed in linear form $x\pm\sqrt{y}$, where x and y are rational, then we have the following trick method to find their values very quickly.

Assume $(a\pm k\sqrt{b})^{1/3}=x\pm\sqrt{y}$, where y contains no square factor. Cubing both sides we have $a\pm k\sqrt{b}=x^3+3xy\pm(3x^2+y)\sqrt{y}$. Hence $\sqrt{b}$ and $\sqrt{y}$ must be similar surds and so $y=b$; from $3x^2+y=k$ we have $x=\{(k-y)/3\}^{1/2}=\{(k-b)/3\}^{1/2}$. If we do not get a rational number for x then $y\neq b$, and y must contain a square factor; we shall discuss this case in detail in the next section.

Example 2. Apply the trick method to solve Example 1.

Solution. Let $(72-32\sqrt{5})^{1/3}=x-\sqrt{y}$, and put $y=5$. Then $x=\{(32-5)/3\}^{1/2}=\sqrt{q}=3$, a rational number. Therefore $(72-32\sqrt{5})^{1/3}=3-\sqrt{5}$, which is the same result as before.

Example 3. Solve the cubic equation $x^3+3x-14=0$.

Solution. Applying the Cardan Formula we have $D=(-14/2)^2+(3/3)^3=50$, $A=(7+\sqrt{50})^{1/3}=(7+5\sqrt{2})^{1/3}=m+\sqrt{n}$, say, and put $n=2$. Then $m=\{(5-2)/3\}^{1/2}=1$. Hence $A=1+\sqrt{2}$, and so $B=(7-\sqrt{50})^{1/3}=1-\sqrt{2}$. Therefore the original roots are

$$x_1 = A + B = (1+\sqrt{2}) + (1-\sqrt{2}) = 2,$$
$$x_2 = A\omega + B\omega^2 = (1+\sqrt{2})\omega + (1-\sqrt{2})\omega^2 = -1 + i\sqrt{6},$$
$$x_3 = A\omega^2 + B\omega = -1 - i\sqrt{6},$$

i.e., the three roots are 2, $-1\pm i\sqrt{6}$.

Example 4. Find the cube root of $38\sqrt{14}-100\sqrt{2}$.

Solution. In this example the terms are all surds; we must factor out one surd and reduce to the form of the preceding example, and then solve it. Here

$$(38\sqrt{14} - 100\sqrt{2})^{1/3} = -\sqrt{2}(50 - 19\sqrt{7})^{1/3}.$$

Now let $(50-19\sqrt{7})^{1/3}=x-\sqrt{y}$, and put $y=7$; then $x=\{(19-7)/3\}^{1/2}=2$. Therefore the required root is $-\sqrt{2}(2-\sqrt{7})=\sqrt{14}-2\sqrt{2}$. Check: $(\sqrt{14}-2\sqrt{2})^3=14\sqrt{14}-84\sqrt{2}+24\sqrt{14}-16\sqrt{2}=38\sqrt{14}-100\sqrt{2}$, and so the result is correct.

Example 5. Find $(54\sqrt{3}+41\sqrt{5})^{1/3}$.

Solution. As above we first transform the given expression to obtain $\sqrt{3}(18+41/3\sqrt{5/3})^{1/3}$. Now let $(18+41/3\sqrt{5/3})^{1/3}=x+\sqrt{y}$, and put $y=5/3$. Then $x=\{\frac{1}{3}(41/3-5/3)\}^{1/2}=2$. Hence the required root is $\sqrt{3}(2+\sqrt{5/3})=2\sqrt{3}+\sqrt{5}$. The check is left to the reader.

II. Now we shall discuss the case where $(a\pm k\sqrt{b})^{1/3}=x\pm\sqrt{y}$, where x and y are rational but y contains a square factor. In this case the above method cannot be used directly, for y may not be equal to b. If we wish to have y equal to b always, we may assume that $(a\pm k\sqrt{b})^{1/3}=x\pm m\sqrt{b}$, where b does not contain a square factor. Cubing both sides we obtain $a\pm k\sqrt{b}=x^3\pm 3x^2m\sqrt{b}+3xm^2b\pm m^3b\sqrt{b}$. Equating the rational parts we obtain $a=x^3+3xm^2b$, and hence

$$m = \left\{\left(\frac{a}{x} - x^2\right)\Big/ 3b\right\}^{1/2}. \tag{3}$$

If a, b, x, m are all integers then from the above expression it may be seen that x must be a factor of a, and thus a/x and x are complementary factors of a. Therefore we may resolve a into complementary factors in all possible ways, and select the factors to satisfy the following conditions:

(1) the larger factor is greater than the square of the smaller one;

(2) the difference between the larger factor and the square of the smaller one is divisible by $3b$.

The value of x can be determined by these two conditions, and the value of m can be found immediately by (3).

Example 6. Find the cube root of $99-70\sqrt{2}$.

Solution. $99=99\times 1=33\times 3=11\times 9$; there are three pairs of complementary factors. But the last pair cannot satisfy the condition (1); therefore we reject it. Next, since $99-1^2=98$ is not divisible by 3×2, it is rejected also. Finally, $33-3^2=24=(3\times 2)\times 2^2$; therefore $x=3$ and $m=2$. Hence $\sqrt[3]{(99-70\sqrt{2})}=3-2\sqrt{2}$.

Alternate solution. We may use the preceding method with a slight modification. First, we assume $\sqrt[3]{(99-70\sqrt{2})}=x-\sqrt{y}$. As we put $y=2$ we have $x=\sqrt{(70-2)/3}=\sqrt{68/3}$ which is not a rational number. Hence the preceding method cannot be used directly.

Secondly, we transform the surd $70\sqrt{2}$ into $35\sqrt{8}$ and then put $y=8$. We get $x=\sqrt{(35-8)/3}=\sqrt{27/3}=3$, a rational number. Therefore $\sqrt[3]{(99-70\sqrt{2})}=3-\sqrt{8}=3-2\sqrt{2}$ as before.

III. Finally, we shall discuss the cube root of $h\sqrt{a}\pm k\sqrt{b}$. Let

$$\sqrt[3]{(h\sqrt{a} \pm k\sqrt{b})} = lx \pm m\sqrt{y},$$

where a, b, x, y are all rational numbers. If they do not contain square factors, then we must have $x=a$, $y=b$, so that

$$h\sqrt{a} \pm k\sqrt{b} = (l\sqrt{a} \pm m\sqrt{b})^3 = l^3a\sqrt{a} \pm 3l^2am\sqrt{b} + 3lm^2b\sqrt{a} \pm m^3b\sqrt{b}.$$

Equating the coefficients of $\sqrt{a}$, we obtain $h=l^3a+3lm^2b$ and hence

$$m = \sqrt{\left(\left(\frac{h}{l} - l^2a\right) \Big/ 3b\right)}. \tag{4}$$

If a, b, h, l, m are all integers, from this expression we know that l must be a factor of h; so we may factor h into all possible pairs of complementary factors, and select the factorization satisfying the following conditions:

(1) the larger factor must be greater than the product of a and the square of the smaller one;

(2) the difference between the larger factor and the product of a and the square of the smaller one is divisible by $3b$.

Example 7. Find $\sqrt[3]{(54\sqrt{3}+41\sqrt{5})}$.

Solution. This example has been solved as Example 5. Here we use the alternate method to solve it. We have $54=54\times 1=27\times 2=18\times 3=9\times 6$. The last two pairs of factors 18, 3; 9, 6 do not satisfy condition (1). Again, $54-3\times 1^2=51$ is not divisible by 3×5, and $27-3\times 2^2=15=3\times 5$. Therefore $l=2$ and $m=1$. Hence we have $\sqrt[3]{(54\sqrt{3}+41\sqrt{5})}=2\sqrt{3}+\sqrt{5}$ as before.

But if a is a multiple of b, then from (4) we know that h must also be a multiple of b, so that we may put $a=a_1b$, $h=h_1b$, where a_1 and h_1 are integers. Then (4) becomes

$$m = \sqrt{\left(\left(\frac{h_1}{l} - l^2a_1\right) \Big/ 3\right)}.$$

Hence the conditions mentioned above can be written as: resolve h_1 into two factors, so that the difference between the larger factor and the product of a_1 and the square of the smaller factor is exactly equal to three times a perfect square.

Example 8. Find $\sqrt[3]{(38\sqrt{14}-100\sqrt{2})}$.

Solution. This has been solved in Example 4. Now we use the alternate method to solve it. Here $a_1=14\div 2=7$, $h_1=38\div 2=19$, and 19 has only one pair of complementary factors, i.e., 19×1; also $19-(1^2\times 7)=12=3\times 2^2$. Therefore $l=1$ and $m=2$. Hence $\sqrt[3]{(38\sqrt{14}-100\sqrt{2})}=\sqrt{14}-2\sqrt{2}$ as before.

References

1. G. Chrystal, Algebra, vol. I, 6th ed., Chelsea, New York, 1959, p. 219.

2. H. S. Hall and S. R. Knight, Higher algebra, 4th ed., St. Martin's, New York, 1932, pp. 70–71.

FUNCTIONS OF A DUAL OR DUO VARIABLE

M. A. B. DEAKIN, University of Chicago

1. Introduction. A striking resemblance exists between the complex numbers $a+ib$, for which $i^2=-1$, the dual numbers $a+\epsilon b$, where $\epsilon^2=0$, and the numbers $a+\omega b$, for which $\omega^2=+1$. This third system, lacking a name, may conveniently be dubbed the system of duo numbers [3]. The algebra of dual numbers is available in the literature [1, 2, 3] and is very similar to the algebra of complex numbers. The algebra of duo numbers exhibits a like similarity. The main differences are:

(i) In the dual number-system, division by ϵ is undefined,

(ii) In the duo number-system, division by $1\pm\omega$ is undefined, whereas no such restrictions exist for complex numbers.

In view of the close parallel between the algebraic properties of these two systems and those of the complex numbers, it is an interesting exercise to attempt to derive analogues of the basic theorems of complex variable theory in these two systems.

2. Dual analysis. For the dual case, use an Argand representation as for the complex numbers, and define:

$$|z| = |x+\epsilon y| = \begin{cases} \sqrt{(x^2+y^2)} & \text{unless} \quad x=y=0 \\ 0 & \text{if} \quad x=y=0. \end{cases} \tag{1}$$

The theorem $|z_1z_2|=|z_1|.|z_2|$, true for the complex numbers, does not hold for duals, but the inequality

$$|z_1z_2| \leqq \sqrt{2}\,|z_1|.|z_2| \tag{2}$$

may be proved by expansion and use of Cauchy's inequality. The other theorems on modulus hold in the dual system.

From this, it follows that convergence, as defined in the theory of complex numbers, may be defined, *mutatis mutandis*, in this system and exact analogues of the basic convergence theorems follow. Thus, continuity may be defined and the basic continuity theorems may be proved in complete analogy to the complex case with the one exception that if $\lim_{z\to z_0} g(z)$ is pure dual, then

$$\lim_{z\to z_0} f(z)/g(z) \text{ does not exist.}$$

These definitions may be used to define a derivative in the usual way, except that the clause is added that the path of approach to the limit is not to include more than a finite number of pure duals $z-z_0$. The laws of differentiation, including the chain rule, are then found to hold. They can be proved in analogy to the complex case.

If we set:

$$f(z) = u(x, y) + \epsilon v(x, y) \tag{3}$$

and require $f(z)$ to be differentiable, u and v are found to be necessarily continuous, and the derivation of the Cauchy-Riemann equations may be followed to give:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \qquad \frac{\partial u}{\partial y} = 0. \tag{4}$$

It thus follows that a differentiable function has the form

$$f(z) = u(x) + \epsilon\{yu'(x) + \phi(x)\}. \tag{5}$$

(Equations (4), (5) are identical with those found for a limited class by E. Study [3]. Equation (5) gives $f(x)=u(x)+\epsilon\phi(x)$ so that Brand's procedure [1] of defining $f(x+\epsilon y)=f(x)+\epsilon yf'(x)$ (a formal Taylor series in ϵy) is seen to give the result (5).)

Integration is defined on rectifiable curves as in complex variable theory, and if $f(z)$ is continuous on the rectifiable curve C, then $\int_C f(z)dz$ exists. Further

$$\left|\int_C f(z)dz\right| \leqq \sqrt{2}\int_\alpha^\beta |f(z(s))|\, ds, \tag{6}$$

where α, β are the initial and final values respectively of the curve parameter s, (arc-length). The theorem is not generally true if the factor $\sqrt{2}$ is omitted, (e.g., integrate $x+\epsilon y$ along $x=y$ from $x=0$ to $x=1$).

If the differentiable function $f(z)$ is integrated around a closed curve C, the result is

$$\begin{aligned}\oint_C f\cdot(z)dz &= \oint_C (u+\epsilon v)(\dot{x}+\epsilon\dot{y})dt \qquad \left(\cdot = \frac{d}{dt}\right)\\ &= \oint_C (u+\epsilon\phi)dx + \epsilon\oint_C (udy + yu'dx)\\ &= \epsilon\oint_C d(yu) = 0.\end{aligned}$$

It is impossible to obtain an analogue of Cauchy's integral formula as division by ϵ is undefined, nor can isolated singularities exist by virtue of equation (5). It is possible to show, however, that $f(z)$ is a Taylor series in z if u, ϕ are Taylor series in x and conversely. The converse is the only part of this theorem that offers any difficulty, and this may be proved by equating real and dual parts of $f(x)=u(x)+\epsilon\phi(x)$.

Not all differentiable functions are representable as power series, however, since the function

$$f(z) = \begin{cases} z^3 & \text{if} \quad x > 0\\ 0 & \text{otherwise}\end{cases}$$

is everywhere differentiable but possesses no power series expansion valid in the neighborhood of the origin. It does follow, however, from (5) that $u(x)$ must be twice differentiable if $f(z)$ is to possess a derivative, and the relation

$$f'(z) = \frac{\partial u}{\partial x} + \epsilon\frac{\partial v}{\partial x},$$

which follows from the proof of (4), guarantees this. This is the analogue of the theorem for complex variable that if $f'(z)$ exists in a domain, then so do all higher derivatives.

3. Duo analysis. Exactly the same process may be applied to the duo numbers. Define

(7) $|z| = |x + \omega y| = \surd(x^2 + y^2)$ (where $\surd$ signifies real positive or zero).

The form of (2) is preserved and all other theorems on modulus hold. Convergence is defined as in the complex case, as is continuity, and the standard theorems on limits from complex number theory apply also in this case, except that the quotient theorem has the different restriction that $\lim_{z\to z_0} g(z)$ is not to be a multiple of $1 \pm \omega$.

Following the same program, we find in analogy to the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{8}$$

so that

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) u = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) v = 0 \tag{9}$$

provided that u, v are sufficiently often differentiable.

Equations (8) give

$$\frac{\partial(u+v)}{\partial x} = \frac{\partial(u+v)}{\partial y}; \qquad \frac{\partial(u-v)}{\partial x} = -\frac{\partial(u-v)}{\partial y}.$$

Thus $u+v = 2v_2(x+y)$, $u-v = -2v_1(x-y)$ where v_2, v_1 may be any differentiable functions. Then

$$f(z) = \{-v_1(x-y) + v_2(x+y)\} + \omega\{v_1(x-y) + v_2(x+y)\}. \tag{10}$$

This equation is analogous to equation (5). An analytic function has conjugate harmonic functions as its real and imaginary parts. Equations (5) and (10) provide the corresponding restrictions of form in the cases which are under review here.

The program may then be continued in complete analogy to that followed in Section 2, and the form of (6) is preserved. Integrating around a closed curve C gives:

$$\oint_C f(z)dz = \oint_C (u + \omega v)(\dot{x} + \omega\dot{y})dt \qquad \left(\cdot = \frac{d}{dt}\right)$$

$$= \oint_C \{-v_1(x-y) + v_2(x+y)\}dx + \oint_C \{v_1(x-y) + v_2(x+y)\}dy$$

$$+ \omega\oint_C \{v_1(x-y) + v_2(x+y)\}dx + \omega\oint_C \{-v_1(x-y) + v_2(x+y)\}dy$$

$$= -\oint_c v_1(x-y)d(x-y) + \oint_c v_2(x+y)d(x+y)$$
$$- \omega\oint_c v_1(x-y)d(x-y) + \omega\oint_c v_2(x+y)d(x+y)$$
$$= 0.$$

The Cauchy integral formula does not hold, as is seen by taking $f(z) \equiv 1$ and $z_0 = 0, f(z_0) = 1$. $\oint_c(dz/z)$ may be calculated where C is the unit circle about the origin. While a Riemann Integral does not exist, due to the lack of definition of $(1 \pm \omega)^{-1}$, the expression has a Cauchy Principal Value of 0.

Neither does equation (10) permit of isolated singularities, for this would imply that either $-v_1+v_2$ or v_1+v_2 failed to exist at the isolated point x_0, y_0. Thus either v_1 or v_2 fails to exist at this point. Suppose, for definiteness, that it is v_1. Then all other x, y on the line $x-y=x_0-y_0$ will also be singularities.

4. Taylor Expansions in Duo Analysis.

LEMMA. $(1+\omega)(x+y)^n = (1+\omega)(x+\omega y)^n$. The proof is inductive; for $n=1$, the statement above reduces to $(1+\omega)(x+y) = (1+\omega)(x+\omega y)$ which may be verified by expansion. (It holds trivially for $n=0$.)

Suppose now that for some k, $(1+\omega)(x+y)^k = (1+\omega)(x+\omega y)^k$. Then

$$\begin{aligned}(1+\omega)(x+y)^{k+1} &= (1+\omega)(x+y)(x+y)^k \\ &= (1+\omega)(x+\omega y)(x+y)^k \quad \text{(by above)} \\ &= (1+\omega)(x+\omega y)(x+\omega y)^k = (1+\omega)(x+\omega y)^{k+1}\end{aligned}$$

whence the result. Similarly $(-1+\omega)(x-y)^n = (-1+\omega)(x-\omega y)^n$. (This result highlights the useful interpretation of ω as the symbol $\pm$.)

THEOREM. *If v_1, v_2 possess Taylor expansions in $(x-y)$, $(x+y)$ respectively, then $f(z)$ possesses a Taylor expansion in z and conversely.*

Proof.

$$\begin{aligned}f(z) &= (-v_1+v_2) + \omega(v_1+v_2) \\ &= \sum_{n=0}^{\infty}\left\{a_n(-1+\omega)(x-y)^n + b_n(1+\omega)(x+y)^n\right\} \quad \text{(by hypothesis)} \\ &= \sum_{n=0}^{\infty}\left\{a_n(-1+\omega)(x+\omega y)^n + b_n(1+\omega)(x+\omega y)^n\right\} \quad \text{(by lemma)} \\ &= \sum_{n=0}^{\infty}\left\{(-a_n+b_n) + \omega(a_n+b_n)\right\}z^n \\ &= \sum_{n=0}^{\infty} c_n z^n\end{aligned}$$

which is a Taylor series in z. Conversely, if a Taylor series is known for $f(z)$, we may reverse the argument above by solving the equations

$$(-a_n + b_n) + \omega(a_n + b_n) = c_n$$

to find Taylor expansions for v_1, v_2.

In general, however, $f(z)$ does not possess a Taylor expansion, as may be seen by considering the function

$$f(z) = \begin{cases} (1-\omega)z^2 & \text{for} \quad y > x \\ 0 & \text{for} \quad y \leqq x. \end{cases}$$

This is differentiable everywhere, but has no Taylor expansion valid in any region about the origin. The above example also shows that $f''(z)$ need not exist even if $f(z)$ is differentiable, nor does v_1 need to be twice differentiable, and the same holds true for v_2.

References

1. L. Brand, Vector and tensor analysis, Wiley, New York, 1947, Ch. 2.
2. M. Eastham, On the definition of dual numbers, Math. Gaz. 45 (1961) p. 232.
3. E. Study, Geometrie der Dynamen, Leipzig, 1903, p. 205, et seq.

SQUARE ROOTS BY AN ITERATIVE METHOD AND ITS GENERALIZATION TO POSITIVE INTEGRAL ROOTS OF ORDER *n*

TOM R. G. BAUEROCHSE, Adelaide Teachers College, Adelaide, South Australia

The aim of this paper is to derive an iterative process which converges to the positive square root of some given positive real number x, and to generalize this iteration to one which will converge to the positive nth root (n integral) of some given positive real x. We shall suppose in the first place, that we require the square root of x by iteration. We use real, positive a, b (not necessarily integers) such that $b^2 > x$, as initial approximations to the root required and write

(1) $$[a - \alpha(n)][b - \beta(n)] = x, \qquad n = 1, 2, \cdots,$$

and suppose that for some N sufficiently large $a - \alpha(N) = b - \beta(N) = \sqrt{x}$ to some desired accuracy. Denoting $\alpha(N)$ by α and $\beta(N)$ by β, we have $a - \alpha = b - \beta$ whence $\alpha = a - b + \beta$.

Suppose also that α, β are such that $|\alpha\beta| \ll 1$, and can be neglected. Then we have

$$(a - \alpha)(b - \beta) = x$$

$$ab - b\alpha - a\beta = x$$

$$ab - b(a - b + \beta) - a\beta = x$$

$$b^2 - (a + b)\beta = x$$

$$(-a_n + b_n) + \omega(a_n + b_n) = c_n$$

to find Taylor expansions for v_1, v_2.

In general, however, $f(z)$ does not possess a Taylor expansion, as may be seen by considering the function

$$f(z) = \begin{cases} (1-\omega)z^2 & \text{for} \quad y > x \\ 0 & \text{for} \quad y \leqq x. \end{cases}$$

This is differentiable everywhere, but has no Taylor expansion valid in any region about the origin. The above example also shows that $f''(z)$ need not exist even if $f(z)$ is differentiable, nor does v_1 need to be twice differentiable, and the same holds true for v_2.

References

1. L. Brand, Vector and tensor analysis, Wiley, New York, 1947, Ch. 2.
2. M. Eastham, On the definition of dual numbers, Math. Gaz. 45 (1961) p. 232.
3. E. Study, Geometrie der Dynamen, Leipzig, 1903, p. 205, et seq.

SQUARE ROOTS BY AN ITERATIVE METHOD AND ITS GENERALIZATION TO POSITIVE INTEGRAL ROOTS OF ORDER n

TOM R. G. BAUEROCHSE, Adelaide Teachers College, Adelaide, South Australia

The aim of this paper is to derive an iterative process which converges to the positive square root of some given positive real number x, and to generalize this iteration to one which will converge to the positive nth root (n integral) of some given positive real x. We shall suppose in the first place, that we require the square root of x by iteration. We use real, positive a, b (not necessarily integers) such that $b^2 > x$, as initial approximations to the root required and write

$$[a - \alpha(n)][b - \beta(n)] = x, \qquad n = 1, 2, \cdots, \tag{1}$$

and suppose that for some N sufficiently large $a - \alpha(N) = b - \beta(N) = \sqrt{x}$ to some desired accuracy. Denoting $\alpha(N)$ by α and $\beta(N)$ by β, we have $a - \alpha = b - \beta$ whence $\alpha = a - b + \beta$.

Suppose also that α, β are such that $|\alpha\beta| \ll 1$, and can be neglected. Then we have

$$\begin{aligned} (a-\alpha)(b-\beta) &= x \\ ab - b\alpha - a\beta &= x \\ ab - b(a - b + \beta) - a\beta &= x \\ b^2 - (a+b)\beta &= x \end{aligned}$$

where

$$b - \beta = b - \frac{b^2 - x}{a + b} = \frac{ab + x}{a + b} = a - \alpha.$$

Denoting $a-\alpha(n)$ by a_{n+1}, $a-\alpha(n-1)$ by a_n, and $a_1=a$, we can write

$$a_{n+1} = (ba_n + x)/(b + a_n), \qquad n = 1, 2, \cdots \tag{2}$$

and this is the required iteration. We need to prove convergence and that the limit is in fact the square root of x.

We have $y=F(y)=(by+x)/(b+y)$ and, for convergence, it is a well-known result [1] that it is necessary that $|F'(y)|<1$ at the root. Clearly, $F'(y)=(b^2-x)/(b+y)^2$ and, substituting $y=\sqrt{x}$, we arrive at the condition $b>-\sqrt{x}$ for convergence and since b is chosen positive and $b^2>x$ the iterates will always converge, independently of the initial choice of a. Clearly, putting $b=a_n$ for all n, we shall have, from (2)

$$a_{n+1} = \frac{a_n^2 + x}{2a_n} = \frac{1}{2}\left(a_n + \frac{x}{a_n}\right) \tag{3}$$

which is a well-known sequence having limit $\sqrt{x}$, and which is derived from Newton's Rule

$$a_{n+1} = a_n - \frac{a_n^2 - x}{2a_n}. \tag{4}$$

The only essential difference between the iterations (2) and (4) lies in their rates of convergence (or order; Newton's Rule is second order and this is independent of the starting point). The rate of convergence of $(ba_n+x)/(b+a_n)$ will depend heavily on $|b-\sqrt{x}|$ since b is unaltered throughout the iteration. Since we know that (2) converges, we know that a limit exists and also that it is unique. We try $\sqrt{x}$, and obtain

$$\frac{b\sqrt{x} + x}{b + \sqrt{x}} = \frac{\sqrt{x}(b + \sqrt{x})}{(b + \sqrt{x})} = \sqrt{x},$$

which is sufficient evidence that the limit is $\sqrt{x}$. The following results were obtained on the IBM1620, using ten iterations.

Clearly, the maximum error occurs in Example 9 where it is 10^{-6}, but here one of the first approximations is equal to x itself. In many instances, e.g., 1 and 4 of this Table, the approximations are so near the required answer that the number or iterations needed to produce the answer given is probably less than ten.

We can construct a similar iteration converging to the cube root of x. This time we shall need three initial approximations a, b, c and we write, as before

$$[a - \alpha(n)][b - \beta(n)][c - \gamma(n)] = x, \qquad n = 1, 2, 3, \cdots \tag{5}$$

where for some N sufficiently large $a-\alpha(N)=b-\beta(N)=c-\gamma(N)=\sqrt[3]{x}$ and, in

TABLE I

	x	b	$a_1=a$	a_{11}	$\sqrt{x}$ (exact)
1.	2.0	1.4150	1.4140	1.41421350	1.41421350
2.	2.0	2.0	2.0	1.41421350	1.41421350
3.	3.0	1.80	1.70	1.73205070	1.73205080
4.	3.0	3.0	3.0	1.73205070	1.73205080
5.	4.0	2.50	1.50	2.00000000	2.00000000
6.	4.0	4.0	4.0	1.99999990	2.00000000
7.	9.0	69.7	52.3	2.99999990	3.00000000
8.	9.0	100.0	100.0	2.99999980	3.00000000
9.	1339.25	0.0	1339.25	36.59678800	36.59678900
10.	10^6	0.0	10^6	10^3	10^3

the limit, we neglect all products involving α, β, γ. Then we have $abc-x=\alpha bc+\beta ac+\gamma ab$ and, denoting $\alpha(N)$, $\beta(N)$, $\gamma(N)$ by α, β, γ respectively, we have $a-\alpha=b-\beta=c-\gamma$ so that $\alpha=a-c+\gamma$, $\beta=b-c+\gamma$. Substituting these in (5) above we obtain an expression for γ, namely

$$\gamma = \frac{bc^2 + ac^2 - abc - x}{ab + ac + bc}$$

whence

$$c - \gamma = \frac{2abc + x}{ab + ac + bc}.$$

We now have the functional form

$$F(a, b, c) = \frac{2abc + x}{ab + ac + bc}. \tag{6}$$

To produce convergence, we need to modify the previous approach ($n=2$) somewhat. Indeed, the following is necessary for all cases with $n>2$. For the case $n=3$, we are seeking the cube root of x. Denote the three initial approximations by t_{11}, t_{21}, t_{31}, for convenience, to give

$$F(t_{11}, t_{21}, t_{31}) = \frac{2t_{11}t_{21}t_{31} + x}{t_{11}t_{21} + t_{11}t_{31} + t_{21}t_{31}}$$

which is of the form

$$t_{i,j+1} = \frac{2C_{ij}t_{ij} + x}{D_{ij}t_{ij} + C_{ij}}, \qquad i = 1, 2, 3, \qquad j = 1, 2, 3, 4, \cdots. \tag{7}$$

Begin with $i=1$, $j=1$. Then $C_{11}=t_{21}t_{31}$, $D_{11}=t_{21}+t_{31}$ so that

$$t_{12} = \frac{2C_{11}t_{11} + x}{D_{11}t_{11} + C_{11}}.$$

We now have a new set of three approximations t_{12}, t_{21}, t_{31}. Now put $i=2$, $j=1$ in (7) to produce t_{22} and then the approximations are t_{12}, t_{22}, t_{31}; now put $i=3$, $j=1$ in (7) to produce t_{32} and then the approximations are t_{12}, t_{22}, t_{32} which is a completely new set of numbers. Continue on in this fashion, at each step calculating new coefficients C_{ij} and D_{ij} given by

$$C_{ij} = \left(\prod_{\substack{k=1 \\ k\neq i}}^{3} t_{kj} \right); \qquad D_{ij} = \sum_{\substack{s=1 \\ s\neq i}}^{3} \left(\prod_{\substack{k=1 \\ k\neq s \\ k\neq i}}^{3} t_{kj} \right).$$

The procedure then is to put $(i, j) = (1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2), \cdots, (s, u-1)$ until $(t_{su})^3 = x$ to some desired accuracy. The process is perhaps best seen through an example. Here $t_{17} = t_{27} = t_{37} = 1.260$ (to 3 decimal places) which

TABLE II, t_{ij}

Cube Root of 2.0

$j \backslash i$	1	2	3
1	1.000	1.000	1.000
2	1.333	1.000	1.000
3	1.333	1.273	1.000
4	1.333	1.273	1.253
5	1.260	1.273	1.253
6	1.260	1.260	1.253
7	1.260	1.260	1.260

is the tabulated four-figure value of the cube root of 2.0. We can stop the iteration when $(t_{su})^3 = x$, or, alternatively, when $t_{1u} = t_{2u} = t_{3u} = k$, for then we would have

$$k = \frac{2k^3 + x}{3k^2}, \qquad \text{i.e.,} \qquad k^3 = x$$

whence we have reached the desired result. The following two tables show results obtained using this iteration procedure to find cube roots ($n=3$) Table III, and fifth roots ($n=5$) Table IV.

The fifth root of 1024.0 was obtained from two different sets of initial approximations,

(i) 4.0, 4.9, 3.01, 2.01, 3.92.
(ii) 1024.0, 1024.0, 1024.0, 1024.0, 1024.0.

The first set required five steps to produce the answer 4.000000 while the second set, where all five approximations equalled 1024.0, required something like 86 (!) steps to produce the same answer, 4.000000. Needless to say, all calculations were done on the computer.

TABLE III

x	t_{su}	$(t_{su})^3$
2.000000	1.259921	1.999999
8.000000	2.000000	8.000000
25.000000	2.924018	24.999999
10.000000	2.154435	10.000000
1.562000	1.160273	1.561999
29.000000	3.072317	28.999999
41.000000	3.448217	40.999999

TABLE IV

x	t_{su}	$(t_{su})^5$
32.000000	2.000000	32.000000
243.000000	3.000000	243.000000
2.000000	1.148698	1.999999
8.000000	1.515717	8.000000
107.000000	2.546108	107.000000
1024.000000	4.000000	1024.000000

Obviously, the foregoing can be generalized to the case when the nth root of x is required. We use n initial approximations $t_{11}, t_{21}, \cdots, t_{n1}$, and we have

$$t_{i,j+1} = \frac{(n-1)C_{ij}t_{ij} + x}{D_{ij}t_{ij} + C_{ij}}, \qquad i = 1, 2, \cdots n, \qquad j = 1, 2, \cdots,$$

and

$$C_{ij} = \prod_{\substack{k=1\\k\neq i}}^{n} t_{kj}, \qquad D_{ij} = \sum_{\substack{s=1\\s\neq i}}^{n} \left(\prod_{\substack{k=1\\k\neq s\\k\neq i}}^{n} t_{kj} \right).$$

I should like to acknowledge the invaluable assistance given me by my friend Mr. Robert P. Backstrom in programming the data necessary for the results given in Tables I, III and IV.

Reference

1. National Physical Laboratory, Teddington, England, Modern computing methods, Notes on Applied Science #16, Her Majesty's Stationery Office, London, 2nd ed., 1962.

THE INTEGRATION OF INVERSE FUNCTIONS

JOHN H. STAIB, Drexel Institute of Technology

Let f and g be inverse functions. It is often the case that an antiderivative for g is more accessible than one for f. Thus it would be nice to be able to replace

TABLE III

x	t_{su}	$(t_{su})^3$
2.000000	1.259921	1.999999
8.000000	2.000000	8.000000
25.000000	2.924018	24.999999
10.000000	2.154435	10.000000
1.562000	1.160273	1.561999
29.000000	3.072317	28.999999
41.000000	3.448217	40.999999

TABLE IV

x	t_{su}	$(t_{su})^5$
32.000000	2.000000	32.000000
243.000000	3.000000	243.000000
2.000000	1.148698	1.999999
8.000000	1.515717	8.000000
107.000000	2.546108	107.000000
1024.000000	4.000000	1024.000000

Obviously, the foregoing can be generalized to the case when the nth root of x is required. We use n initial approximations $t_{11}, t_{21}, \cdots, t_{n1}$, and we have

$$t_{i,j+1} = \frac{(n-1)C_{ij}t_{ij} + x}{D_{ij}t_{ij} + C_{ij}}, \qquad i = 1, 2, \cdots n, \qquad j = 1, 2, \cdots,$$

and

$$C_{ij} = \prod_{\substack{k=1 \\ k\neq i}}^{n} t_{kj}, \qquad D_{ij} = \sum_{\substack{s=1 \\ s\neq i}}^{n} \Bigg(\prod_{\substack{k=1 \\ k\neq s \\ k\neq i}}^{n} t_{kj} \Bigg).$$

I should like to acknowledge the invaluable assistance given me by my friend Mr. Robert P. Backstrom in programming the data necessary for the results given in Tables I, III and IV.

Reference

1. National Physical Laboratory, Teddington, England, Modern computing methods, Notes on Applied Science #16, Her Majesty's Stationery Office, London, 2nd ed., 1962.

THE INTEGRATION OF INVERSE FUNCTIONS

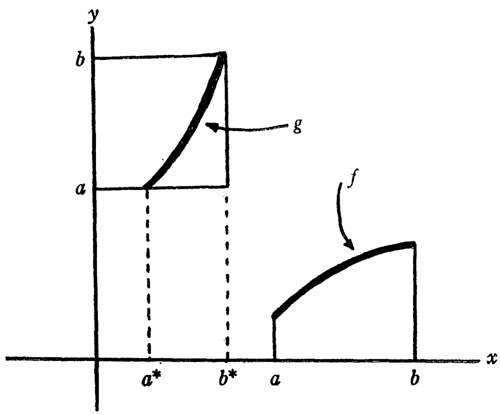
JOHN H. STAIB, Drexel Institute of Technology

Let f and g be inverse functions. It is often the case that an antiderivative for g is more accessible than one for f. Thus it would be nice to be able to replace integrals of f with integrals of g. The following formula does the trick:

$$\int_a^b f(x)dx = (bb^* - aa^*) - \int_{a^*}^{b^*} g(x)dx,$$

where $u^* = f(u)$. We make several observations about this identity.

1. *A pictorial derivation is available.* From the figure it is easy to see that



$$\int_a^b f(x)dx = b^*(b-a) - \int_{a^*}^{b^*} [g(x) - a]dx.$$
$$= \cdots .$$

2. *Its proof provides a nice application for the integration-by-parts formula.* Let $F' = f$; then

$$\int_{a^*}^{b^*} [g(x)\cdot 1]dx = xg(x)\Big|_{a^*}^{b^*} - \int_{a^*}^{b^*} xg'(x)dx$$
$$= b^*b - a^*a - \left[F(g(x))\Big|_{a^*}^{b^*}\right]$$
$$= [bb^* - aa^*] - [F(b) - F(a)].$$

3. *It is instructive.* Suppose that we know an antiderivative for g, say G. Then we may write

$$\int_a^x f(u)du = [xf(x) - af(a)] - \int_{f(a)}^{f(x)} g(u)du$$
$$= xf(x) - G(f(x)) + c_a.$$

Thus we have an analogue for the formula

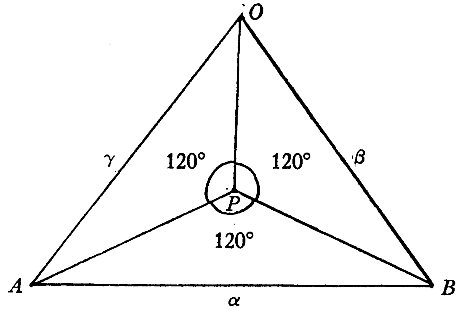
$$f'(x) = \frac{1}{g'(f(x))} \cdot$$

A REMARK ON A NOTE OF S. M. SHAH

S. VENKATRAMAIAH, V.S.R. College, Tenali, India

Let OAB be an acute angled triangle and P a variable point in its plane. Steiner [1] proved that the expression $OP+PA+PB$ is minimum when $\angle OPA = \angle OPB = \angle APB = 120°$, i.e., when each side of the triangle subtends an angle of 120° at the point P.

Professor S. M. Shah in [2] derived an expression for the minimum value of $OP+PA+PB$ in terms of the sides of the triangle OAB. In his article, at one stage, he made use of complex numbers. The aim of the present note is to derive the same expression for $\min(OP+PA+PB)$ by using only elementary trigonometry.



Let $AB=\alpha$, $BO=\beta$, and $OA=\gamma$. Now let P be the position of the variable point for which the expression $OP+PA+PB$ has the minimum value, so that, by Steiner we have $\angle OPA = \angle OPB = \angle APB = 120°$. From ΔOPB we have $AB^2 = AP^2 + PB^2 + 2AP \cdot PB \cdot \cos 120° = AP^2 + PB^2 + AP \cdot PB$, $\therefore AP^2 + PB^2 + 4AP \cdot PB = AB^2 + 3AP \cdot PB = \alpha^2 + 4\sqrt{3}\, \Delta APB$.

Similarly,

$$BP^2 + PO^2 + 4BP \cdot PO = \beta^2 + 4\sqrt{3}\, \Delta BPO$$

$$AP^2 + PO^2 + 4AP \cdot PO = \gamma^2 + 4\sqrt{3}\, \Delta APO.$$

Adding, we get,

$$\begin{aligned} 2(AP + PO + PB)^2 &= \sum \alpha^2 + 4\sqrt{3}\, [\Delta AOB + \Delta POB + \Delta POA] \\ &= \sum \alpha^2 + 4\sqrt{3}\, \Delta AOB \\ &= \sum \alpha^2 \\ &\quad + \sqrt{3}\, \sqrt{((\alpha+\beta+\gamma)(\alpha+\beta-\gamma)(\alpha+\gamma-\beta)(\gamma+\beta-\alpha))} \\ &= \sum \alpha^2 + \sqrt{3}\, \sqrt{(2 \sum \alpha^2\beta^2 - \sum \alpha^4)} \\ &= \sum \alpha^2 + \sqrt{(6 \sum \alpha^2\beta^2 - 3 \sum \alpha^4)} \end{aligned}$$

$$\therefore OP + PA + PB = \sqrt{\tfrac{1}{2}(\sum \alpha^2 + \sqrt{(6 \sum \alpha^2\beta^2 - 3 \sum \alpha^4)})}.$$

References

1. R. Courant and H. Robbins, What is mathematics?, Oxford University Press, New York, 1941.

2. S. M. Shah, A note on Steiner's problem, Math. Student, vol. 29, 1961.

ANSWERS

A388. Consider

$$I = \int \frac{dx}{x + x^m} = \int \frac{x^{-m}dx}{x^{1-m} + 1}$$

$$= \frac{1}{1 - m} \log (x^{1-m} + 1).$$

Thus

$$\int_{\sqrt{2}}^{\infty} \frac{dx}{x + x^{\sqrt{2}}} = (\sqrt{2} - 1) \log [1 + 2^{(1-\sqrt{2})/2}].$$

A389. Set $z = x + iy$ and let R denote the real part of z. Then the given limit may be written

$$\lim_{z \to 0} R\left[e^{-R(z)} \frac{e^z - 1}{z} \right]$$

which has the value 1.

A390. Let Q equal the rationals in $[0, 1]$. The natural map from $[0, 1]$ to $[0, 1]/Q$ is open. This latter space is indiscrete, therefore any map from it is open.

A391. It is sufficient to show that at least one of the integers S, $S+2^k$, $S+2^{k+1}$ is composite. If k is odd, then if $S \equiv 0 \pmod 3$, S is composite. If $S \equiv 1 \pmod 3$ then $S+2^k = 1-1 \equiv 0 \pmod 3$ and $S+2^{k+1}$ is composite. Finally if $S \equiv 2 \pmod 3$, $S+2^{k+1} \equiv 2+1 \equiv 0 \pmod 3$, and $S+2^{k+1}$ is composite. The argument is similar if k is even.

A392. For $x = 1, 2, \cdots, 8$, $x^2 \equiv 11 \pmod{17}$ is invalid. Therefore 11 is not a quadratic residue modulo 17.

DIAGONALIZING POSITIVE DEFINITE MATRICES

JAMES DUEMMEL, University of Montana

A diagonal matrix congruent to a given $n \times n$ symmetric matrix A may be constructed by the following procedure. With respect to the standard basis of n-tuple space, A represents a bilinear form $f(\alpha, \beta)$. Choose a new basis $\alpha_1, \cdots, \alpha_n$ where each α_j is in the null space of the linear functionals $f_i(\beta) = f(\alpha_i, \beta)$ for all i where $1 \leq i < j$. With respect to this new basis, $f(\alpha, \beta)$ will be represented by a diagonal matrix ([1], pp. 131 to 136).

If we regard vectors as $1 \times n$ matrices and denote the transpose of a matrix by a prime, then $f(\alpha, \beta)$ is the single entry in the 1×1 matrix $\alpha A \beta'$. In particular, the linear functional $f_i(\beta)=f(\alpha_i, \beta)$ can be represented by the $1\times n$ matrix $\alpha_i A$. If P is the matrix whose rows are $\alpha_1, \alpha_2, \cdots, \alpha_n$ in order, then PAP' is a diagonal matrix congruent to A. The rows of PA are just $\alpha_1 A, \cdots, \alpha_n A$ in order. These will already have been calculated in determining the α_i.

These procedures combined with the cofactor expansion of a determinant give a fairly simple proof of a standard theorem about positive definite matrices and, for small n, a reasonable method of diagonalizing symmetric matrices.

Let A be an $n\times n$ symmetric matrix and let $A(i)$, $i=1, 2, \cdots, n$ denote the matrix obtained from A by eliminating the first $i-1$ rows and columns. Let $A(i)_{kl}$ be the cofactor of a_{kl} in $A(i)$ and let $d_i=\det A(i)$ for $i=1, 2, \cdots, n$. Thus $d_1=\det A(1)=\det A$, $d_n=a_{nn}$ and $d_{i+1}=A(i)_{ii}$, $i=1, 2, \cdots, n-1$.

THEOREM. *The real symmetric matrix A is positive definite if $d_i>0$ for $i=1, 2, \cdots, n$.*

Proof. Let $\alpha_1=(A(1)_{11}, A(1)_{21}, \cdots, A(1)_{n1})$. Then the functional $f_1(\beta)=f(\alpha_1, \beta)$ is represented by the $1\times n$ matrix $\alpha_1 A=[\det A(1), 0 \cdots 0]$. The null space of $f_1(\beta)$ is the set of all vectors whose first component is zero. We choose $\alpha_2=(0, A(2)_{22}, A(2)_{32}, \cdots, A(2)_{n2})$. The linear functional $f(\alpha_2, \beta)$ is represented by $\alpha_2 A=[k_{12}, \det A(2), 0 \cdots 0]$, where

$$k_{12} = \sum_{i=2}^{n} a_{i1}A(2)_{i2}.$$

The intersection of the null spaces of $f(\alpha_2, \beta)$ and $f(\alpha_1, \beta)$ consists of all vectors whose first two components are zero. We continue this process to define

$$\alpha_i = (0, \cdots, 0, A(i)_{ii}, A(i)_{i+1i}, \cdots, A(i)_{ni}) \quad (1 \leqq i < n)$$
$$\alpha_n = (0, \cdots, 0, 1).$$

Then $f(\alpha_i, \beta)$ is represented by a $1\times n$ matrix whose ith entry is $\det A(i)=d_i$ and whose entries after the ith are zeros. The first $i-1$ entries are combinations of cofactors of $A(i)$.

Let P be the matrix whose rows are $\alpha_1, \alpha_2, \cdots, \alpha_r$ in order. P is a superdiagonal matrix in which the ith diagonal entry is $A(i)_{ii}$ for $1\leqq i<n$ and 1 is the nth diagonal entry. Since $A(i)_{ii}=d_{i+1}>0$, P is nonsingular. From the calculations of $\alpha_i A$ above, it follows that PA is a subdiagonal matrix. The transpose P' of P is also subdiagonal, and therefore PAP' is a subdiagonal matrix. But since PAP' is symmetric, it has zeros below the diagonal also. It follows easily from the form of PA and P' that the diagonal elements are $d_1A(1)_{11}$, $d_2A(2)_{22}, \cdots, d_{n-1}A(n-1)_{n-1\,n-1}, d_n$ or $d_1d_2, d_2d_3, \cdots, d_{n-1}d_n, d_n$ in order. Each of these is positive if $d_i>0$ for all i. This proves the theorem.

Since the nonsingularity of P followed from the fact that $d_2, d_3, \cdots, d_n$ were nonzero, the method can be used to prove more general theorems about the definiteness properties of other than positive definite matrices.

Reference

1. E. D. Nering, Linear Algebra and Matrix Theory, Wiley, New York, 1963.

MAXIMUM AREA OF A REGION BOUNDED BY A CLOSED POLYGON WITH GIVEN SIDES

HUSEYIN DEMIR, Middle East Technical University, Ankara, Turkey

The isoperimetric problem [1] in the calculus of variations suggests naturally the substitution of the closed curve of given perimeter by a closed polygon of given sides, hence of given perimeter. The modified problem becomes an extremal problem for a function of several variables with some constraining relations, and the answer is expected to be a polygon inscribed in a circle. Indeed the following theorem holds.

THEOREM. *The maximum area of the plane region bounded by a simple closed polygon with given sides occurs when the polygon is inscribed in a circle.*

Proof. Let $A_0A_1 \cdots A_nA_0$ be a simple closed polygon having the given sides

$$A_iA_{i+1} = a_i, \qquad a_i > 0, \qquad i = 0, 1, \cdots, n \tag{1}$$

with $A_{n+1} \equiv A_0$. Referring to polar coordinates, let A_0 be the pole and A_0A_1 be the polar axis and let

$$A_i(\theta_i, r_i), \qquad i = 1, \cdots, n \tag{2}$$

be the coordinates of the vertices with $r_i > 0$ and $\theta_1 = 0$ (Figure 1).

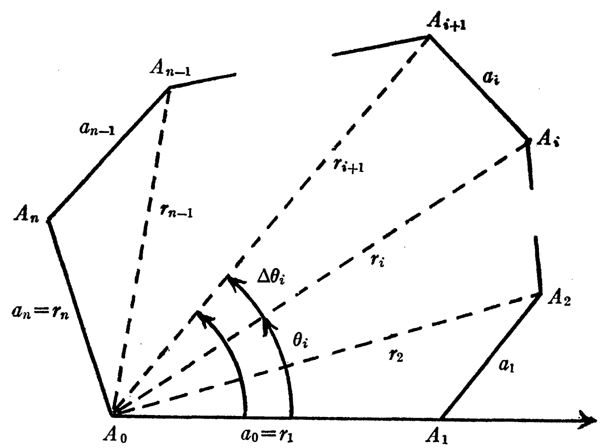


FIG. 1.

The area S of the region bounded by the closed polygon is given by

$$2S = \sum_{i=1}^{n-1} r_i r_{i+1} \sin \Delta\theta_i \tag{3}$$

where

$$\Delta\theta_i = \theta_{i+1} - \theta_i, \qquad i = 1, \cdots, n-1, \tag{4}$$

and the r_i's satisfy the relations

$$g_i \equiv r_i^2 + r_{i+1}^2 - a_i^2 - 2r_i r_{i+1} \cos \Delta\theta_i = 0, \qquad i = 1, \cdots, n-1. \tag{5}$$

Here we have a problem of maximizing or minimizing $4S$ with the constraining relations (5) and the solution is obtained by the method of Lagrange multipliers [2].

Consider

$$f(\theta_2, \cdots, \theta_n; r_2, \cdots, r_{n-1}) = 4S + \sum_{i=1}^{n-1} \lambda_i g_i$$

or, explicitly, the function

$$\begin{aligned} f = {} & 2r_1 r_2 \sin(\theta_2 - \theta_1) + \cdots + 2r_{n-1} r_n \sin(\theta_n - \theta_{n-1}) \\ & + \lambda_1[r_1^2 + r_2^2 - a_1^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)] \\ & + \lambda_2[r_2^2 + r_3^2 - a_2^2 - 2r_2 r_3 \cos(\theta_3 - \theta_2)] \\ & \cdots\cdots\cdots\cdots \\ & + \lambda_{n-1}[r_{n-1}^2 + r_n^2 - a_{n-1}^2 - 2r_{n-1} r_n \cos(\theta_n - \theta_{n-1})] \end{aligned} \tag{6}$$

with λ_i's as Lagrange multipliers and $\theta_1 = 0$, $r_1 = a_0$ and $r_n = a_n$.

First, from $\partial f/\partial \theta_i = 0$, $i = 2, \cdots, n$, we get, using (4), the set of relations

$$\begin{aligned} & (\cos \Delta\theta_1 + \lambda_1 \sin \Delta\theta_1) r_1 - (\cos \Delta\theta_2 + \lambda_2 \sin \Delta\theta_2) r_3 = 0 \\ & \cdots\cdots\cdots\cdots \\ & (\cos \Delta\theta_{n-2} + \lambda_{n-2} \sin \Delta\theta_{n-2}) r_{n-2} - (\cos \Delta\theta_{n-1} + \lambda_{n-1} \sin \Delta\theta_{n-1}) r_n = 0 \\ & (\cos \Delta\theta_{n-1} + \lambda_{n-1} \sin \Delta\theta_{n-1}) = 0. \end{aligned} \tag{7}$$

The last equation of (7) gives λ_{n-1}; the one which precedes it gives λ_{n-2}; and finally the first one λ_1. Hence we have

$$\lambda_i = -\cot \Delta\theta_i, \qquad i = 1, \cdots, n-1 \tag{8}$$

provided $\Delta\theta_i = 0$.

Next, from $\partial f/\partial r_i = 0$, $i = 2, \cdots, n-1$, we get

$$\begin{aligned} & (\sin \Delta\theta_1 - \lambda_1 \cos \Delta\theta_1) r_1 + (\lambda_1 + \lambda_2) r_2 + (\sin \Delta\theta_2 - \lambda_2 \cos \Delta\theta_2) r_3 = 0 \\ & \cdots\cdots\cdots\cdots \\ & (\sin \Delta\theta_{n-2} - \lambda_{n-2} \cos \Delta\theta_{n-2}) r_{n-2} + (\lambda_{n-2} + \lambda_{n-1}) r_{n-1} \\ & \qquad + (\sin \Delta\theta_{n-1} - \lambda_{n-1} \cos \Delta\theta_{n-1}) r_n = 0. \end{aligned} \tag{9}$$

For convenience we write the relations (9) in the form

$$\begin{aligned} & (\sin \Delta\theta_{i-1} - \lambda_{i-1} \cos \Delta\theta_{i-1}) r_{i-1} + (\lambda_{i-1} + \lambda_i) r_i \\ & \qquad + (\sin \Delta\theta_i - \lambda_i \cos \Delta\theta_i) r_{i+1} = 0, \qquad i = 2, \cdots, n-1, \end{aligned} \tag{10}$$

where, by (8), having

$$\sin \Delta\theta_{i-1} - \lambda_{i-1} \cos \Delta\theta_{i-1} = \sin \Delta\theta_{i-1} + \frac{\cos^2 \Delta\theta_{i-1}}{\sin \Delta\theta_{i-1}} = \frac{1}{\sin \Delta\theta_{i-1}}$$

$$\lambda_{i-1} + \lambda_i = -\frac{\cos \Delta\theta_{i-1}}{\sin \Delta\theta_{i-1}} - \frac{\cos \Delta\theta_i}{\sin \Delta\theta_i} = -\frac{\sin(\Delta\theta_i + \Delta\theta_{i-1})}{\sin \Delta\theta_{i-1} \sin \Delta\theta_i}$$

$$\sin \Delta\theta_i - \lambda_i \cos \Delta\theta_i = \frac{1}{\sin \Delta\theta_i}$$

the relations (10) reduce to

$$(11) \quad r_{i-1} \sin \Delta\theta_i - r_i \sin(\Delta\theta_{i-1} + \Delta\theta_i) + r_{i+1} \sin \Delta\theta_{i-1} = 0, \quad i = 2, \cdots, n-1.$$

Hence the relations (11) are ones satisfied by the polygon of maximum or minimum area. We show that such a polygon is an inscribable one. In fact, if the polygon $A_0A_1 \cdots A_n$ is inscribable, it is inverted into a straight line $d: A_1' A_2' \cdots A_n'$ by an inversion with center at the pole A_0 and power equal to 1, say, and the polar coordinates of A_i' are given by

$$(12) \qquad A_i' (\theta_i' = \theta_i, \qquad r_i' = 1/r_i).$$

For a suitable angle θ, the polar equation of the line d being $r' \cos(\theta' - \theta) = \text{constant}$, we have $r_1' \cos(\theta_1 - \theta) = r_2' \cos(\theta_2 - \theta) = \cdots = r_n' \cos(\theta_n - \theta)$ or, by (12),

$$(13) \qquad r_1 : r_2 : \cdots : r_n = \cos(\theta_1 - \theta) : \cos(\theta_2 - \theta) : \cdots : \cos(\theta_n - \theta)$$

with $\theta_1 = 0$.

Now substituting (13) in the left hand side of (11) we have

$$\begin{aligned} u_i &= \cos(\theta_{i-1} - \theta) \sin \Delta\theta_i - \cos(\theta_i - \theta) \sin(\Delta\theta_{i-1} + \Delta\theta_i) \\ &\quad + \cos(\theta_{i+1} - \theta) \sin(\Delta\theta_{i-1}) \\ 2u_i &= 2\cos(\theta_{i-1} - \theta) \sin(\theta_{i+1} - \theta_i) - 2\cos(\theta_i - \theta) \sin(\theta_{i+1} - \theta_{i-1}) \\ &\quad + 2\cos(\theta_{i+1} - \theta) \sin(\theta_i - \theta_{i-1}) \\ &= \sin(\theta_{i-1} - \theta + \theta_{i+1} - \theta_i) - \sin(\theta_{i-1} - \theta - \theta_{i+1} + \theta_i) \\ &\quad - \sin(\theta_i - \theta + \theta_{i+1} - \theta_{i-1}) + \sin(\theta_i - \theta - \theta_{i+1} + \theta_{i-1}) \\ &\quad + \sin(\theta_{i+1} - \theta + \theta_i - \theta_{i-1}) - \sin(\theta_{i+1} - \theta - \theta_i + \theta_{i-1}) \equiv 0. \end{aligned}$$

Hence the polygon $A_0A_1 \cdots A_r$ is inscribed in a circle. Now we need to show that a polygon of given sides can be made inscribable in a circle. Indeed, let C be a circle of sufficiently large radius, and starting with an arbitrary point A_0 on C, consider the chords $A_0A_1, A_1A_2, \cdots, A_nA_{n+1}$ equal respectively to $a_0, a_1, \cdots, a_n$. If A_{n+1} does not coincide with A_0, by decreasing the radius of the circle C continuously one can make A_{n+1} coincide with A_0, and the polygon becomes inscribed in a circle.

To complete the proof of the theorem we show that the inscribable polygon gives the maximum area. This we do by showing that when $\Delta\theta_i = 0$ for some i, we do not have the maximum area.

If the polygon is a triangle, the figure is rigid and there is no problem. We then first consider quadrangles, and prove that a quadrangle $A_0A_1A_2A_3$ with given sides a_0, a_1, a_2, a_3 can be deformed, keeping the sides constant, into a straight line or else to a quadrangle as shown in Figure 3, and in each case some $\Delta\theta_i = 0$.

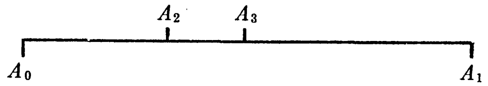


FIG. 2.

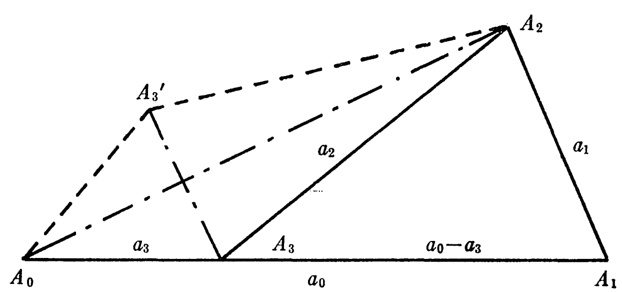


FIG. 3.

Indeed, if the sum of two consecutive sides is equal to the sum of the other two sides then the quadrangle can be deformed into a straight line corresponding to a minimum (zero) area. On the other hand, if the quadrilateral is not collapsed to a straight line to begin with, then the sum of any three sides exceeds the fourth side. Since all four sides are not equal, assume $a_3 > a_0$. Then, $a_0 + a_1 + a_2 > a_3$ and $a_1 + a_2 > a_3 - a_0 = |a_3 - a_0|$ and the quadrilateral can be collapsed into one like that in Figure 3 which does not correspond to maximum area, for, taking the symmetry A_3' of A_3 with respect to A_0A_2 we obtain a quadrilateral of larger area. Hence the theorem is proved for quadrilaterals.

If the polygon is a pentagon, it can be deformed into a quadrangle because there exists at least a pair of consecutive sides such that their sum is less than the sum of the other three sides. This proves the theorem for pentagons.

Now let $A_0A_1 \cdots A_n$ be a polygon with $n > 4$. By an argument similar to one given for pentagons, we can show that it can be deformed into a polygon of n sides and the latter to one with $n-1$ sides and so on. Hence the given polygon can be deformed either into a straight line or else to the simple figure as shown in Figure 3 and, in both cases, some $\Delta\theta_i = 0$ which was ruled out by the condition on (8). Thus the proof of the theorem is completed.

References

1. R. Weinstock, Calculus of variations, McGraw-Hill, New York, 1952, pp. 48–56.
2. I. S. Sokolnikoff, Advanced calculus, McGraw-Hill, New York, 1939, pp. 327–334.

ON THE COMPLETE INDEPENDENCE OF THE AXIOMS OF A SEMINATURAL SYSTEM

SAMUEL T. STERN, State University College at Buffalo

In this paper we shall investigate a set of four postulates characterizing nonempty well-ordered sets without last element, with our attention directed to the concept of complete independence as defined by E. H. Moore, [1].

We shall exhibit a set of models and prove two theorems, demonstrating the consistency and independence of the postulates and the only logical relations that exist between the postulates and their contradictions.

(1) DEFINITION. *A set N, together with a binary relation R on N is said to be a seminatural system if and only if the following postulates are satisfied:*

P-1. *N is simply ordered with respect to R.*

P-2. *N contains an element a, such that there exists no $x \in N$ for which xRa.*

P-3. *For every element x of N, there exists an element y of N, $y \neq x$, such that xRy.*

P-4. *If G is a nonempty subset of N such that for every $x \in N$, $I(x) \subset G$ implies $x \in G$, then $G = N$, where $I(x) = \{y : y \in N \text{ and } yRx\}$.*

The consistency of the postulates of (1) is established by taking as an interpretation the set of natural numbers in their natural order. Another mod e of (1) is the set

$$W = \{0, 1, 2, \cdots, n, \cdots, 0', 1', 2', \cdots, n', \cdots,\}$$

where $j' \neq j$ for all nonnegative integers j; together with the relation $<$ defined as follows:

$j < k'$ for all nonnegative integers j, k;

$j < k$ and $j' < k'$ if and only if the nonnegative integer k exceeds the nonnegative integer j in the natural order of the integers. P-4 is satisfied because W is well-ordered with respect to $<$, [3].

Properties of seminatural systems are discussed in [2].

(2) DEFINITION. *A system of postulates Σ is said to be completely independent if, for every subsystem A of Σ, the system $A \cup [\sim(\Sigma - A)]$ is satisfiable, where $\sim B$ denotes the set of negations of B.*

We shall exhibit for each integer $k = 0, 1, 2, 3, 4$ and for any k of the postulates of (1) an example in which these k postulates are true and the remaining $4-k$ are false; with one exception: P-1, P-4 true and P-2, P-3 false. We exclude the latter for a good reason: P-2 is a consequence of P-1, P-4 and the negation of P-3. Hence, no situation can exist in which P-1, P-4 are true and P-2, P-3 are false.

In each example we shall designate only those postulates which are true, it being understood that the remaining postulates are false.

(3) $A=\varnothing$. Let $N=\{a, b\}$ and define R as follows:

$$a \; R \; a$$

$$b \; R \; b$$

P-1 is false, for R is not a simple order relation on N. P-2 is false also. P-3 is false, since there exists no y distinct from a such that aRy. P-4 is false, for consider the subset $G=\{a\}$. $I(a)=\{a\}$, $I(a)\subset G$, and $a\in G$. $I(b)=\{b\}$ and $I(b)\not\subset G$. Thus for every $x\in N$, $I(x)\subset G$ implies $x\in G$. Nevertheless $G\neq N$. Hence P-4 is false.

(4) $A=\{\text{P-1}\}$. Let $N=\{x: x \text{ is real and } 0<x\leqq 1\}$, where $<$ is the conventional order relation on the real numbers. Let G be the subset of N consisting of the real numbers in the interval $\{x: 0<x\leqq\frac{1}{2}\}$. Then for every $x\in N$, $I(x)\subset G$ implies $x\in G$, for if $x\in N-G$, then $I(x)\not\subset G$. But $G\neq N$. Hence P-4 is false.

(5) $A=\{\text{P-2}\}$. Let $N=\{a, b\}$ and define R as follows:

$$b \; R \; b$$

$$a \; R \; b$$

P-2 is true if we take a as the required element. Let $G=\{a\}$. $I(a)$ is empty. Hence $I(a)\subset G$. But $a\in G$. $I(b)=\{a, b\}$, and $I(b)\not\subset G$. Thus $I(x)\subset G$ implies $x\in G$. However, $G\neq N$.

(6) $A=\{\text{P-3}\}$. Let $N=\{a, b\}$ and

$$b \; R \; b$$

$$a \; R \; b$$

$$b \; R \; a$$

$$a \; R \; a$$

P-2 is false, since for every $x\in N$ there exists $y\in N$ such that yRx. Let $G=\{a\}$. Then, $I(a)=\{a, b\}$, and $I(a)\not\subset G$. $I(b)=\{a, b\}$ and $I(b)\not\subset G$. Hence, for all $x\in N$, $I(x)\subset G$ implies $x\in G$. But $G\neq N$.

(7) $A=\{\text{P-4}\}$. Let $N=\{a\}$ and

$$a \; R \; a$$

P-4 is true, since the only nonempty subset of N is N itself.

(8) $A=\{\text{P-1, P-2}\}$. Let $N=\{x: x \text{ is real and } 0\leqq x\leqq 1\}$, where $<$ represents the conventional ordering of the real numbers. P-4 is false, for we have only to take as $G\{x: 0\leqq x\leqq\frac{1}{2}\}$. Then, for every $x\in N$, $I(x)\subset G$ implies $x\in G$. We see, however, that $G\neq N$.

(9) $A=\{\text{P-1, P-3}\}$. Let $N=\{x: x \text{ is real and } 0<x<1\}$. Again, $<$ represents

the usual ordering of the reals. To show that P-4 is false, we let G be the half-open interval $\{x: 0<x\leqq\frac{1}{2}\}$ and the argument is the same as that of the previous example.

(10) $A=\{\text{P-2, P-3}\}$. Let $N=\{a, b, c\}$ with R defined in the following way:

$$b\ R\ a$$

$$a\ R\ b$$

$$c\ R\ a$$

$$c\ R\ b$$

P-2 is true if we choose c as the desired element. P-4 is false, for we may take as G the subset $\{c\}$. Then, $I(a)=\{b, c\}$, and $I(a)\not\subset G$. $I(b)=\{a, c\}$, and $I(b)\not\subset G$. $I(c)=\emptyset$, $I(c)\subset G$, and $c\in G$. Thus, for every $x\in N$, $I(x)\subset G$ implies $x\in G$. But $G\neq N$.

(11) $A=\{\text{P-2, P-4}\}$. Let the set $N=\{a, b, c\}$ and R:

$$b\ R\ c$$

$$a\ R\ b$$

P-2 is true, for we take a as the desired element. To prove that P-4 holds, we consider every nonempty subset G of N. If $G=\{a\}$, then $I(b)=\{a\}$, $I(b)\subset G$, but $b\notin G$. Hence G does not have to equal N. If $G=\{b\}$ then $I(a)=\emptyset$. $I(a)\subset G$, but $a\notin G$. Again G is not required to equal N. If $G=\{c\}$, then $I(a)=\emptyset$, $I(a)\subset G$ but $a\notin G$. If $G=\{a, b\}$ then $I(c)=\{b\}$, $I(c)\subset G$ but $c\notin G$. If $G=\{a, c\}$ then $I(b)=\{a\}$, $I(b)\subset G$ but $b\notin G$. If $G=\{b, c\}$, then $I(a)=\emptyset$, $I(a)\subset G$ but $a\notin G$. The only remaining subset of N is N itself. Hence P-4 is true. Now P-1 is false, for aRb and bRc, but $a\not R c$. P-3 is false since there exists no $y\in N$ such that cRy.

(12) $A=\{\text{P-3, P-4}\}$. We let $N=\{a, b\}$ and R:

$$a\ R\ b$$

$$b\ R\ a$$

P-4 is true, for if $G=\{a\}$, then $I(b)=\{a\}$, $I(b)\subset G$ but $b\notin G$. On the other hand, if $G=\{b\}$, then $I(a)=\{b\}$, $I(a)\subset G$ but $a\notin G$. Hence the subsets $\{a\}$ and $\{b\}$ are not required to equal N. The only remaining subset is N.

(13) $A=\{\text{P-1, P-2, P-3}\}$. Let $N=\{x: x \text{ is real and } 0\leqq x<1\}$ and $<$ the usual ordering of the reals. P-4 is false, for we have only to take as G the closed interval $\{x: 0\leqq x\leqq\frac{1}{2}\}$. Then, for every $x\in N$, $I(x)\subset G$ implies $x\in G$, but $G\neq N$.

(14) $A=\{\text{P-1, P-2, P-4}\}$. Let $N=\{a\}$ and define the relation R on N as the empty set. Then $a\not R a$. Now P-1 is true. P-2 is true, since we can take a as the required element. P-4 is true, since the only nonempty subset of N is N. P-3 is false, for there exists no $y\in N$ such that aRy.

(15) $A = \{$P-1, P-3, P-4$\}$. Let $N = \varnothing$. We define R as the empty subset of $N \times N$. P-4 is true, since N contains no nonempty subsets. P-2 is false, for N contains no elements whatsoever.

The last example is a rather dull one, since N is the empty set. However, the empty set is the only set for which P-1, P-3, P-4 are true and P-2 is false, as the next theorem shows.

(16) THEOREM. *Let N be a set with a binary relation $<$ which satisfies* P-1 *and* P-4 *but does not satisfy* P-2. *Then N must be empty.*

Proof. Suppose N is not empty. Then N contains an element x. Let G be the subset of N consisting of the element x and those $y \in N$ such that $x < y$. Then G is not empty. Let $q \in N$ and assume that $I(q) \subset G$. Since P-2 is false there exists an element y^* such that $y^* < q$. Hence $y^* \in I(q)$. If $q = x$, then $q \in G$. If $q \neq x$ then consider two cases: (i) where $y^* = x$ and (ii) where $y^* \neq x$. In the first case $x < q$ and hence, $q \in G$. In the second case $x < y^*$, since $y^* \in G$,and hence $x < y^* < q$, which by P-1 implies $x < q$. Hence $q \in G$. By P-4, $G = N$, i.e., for all $y \in N$ such that $y \neq x$, $x < y$. But this makes x the first element of N, contradicting the hypothesis that P-2 is false. Hence N is empty.

(17) $A = \{$P-2, P-3, P-4$\}$. Let N be the set of positive integers and define R on N as follows:

$$mRn \quad \text{if and only if} \quad n = m + 1.$$

Thus, $1R2$ and $2R3$, but $1 \not R 3$. Hence P-1 is false. P-2 is true when we take 1 as the required element. P-3 is true, since for every $m \in N$, there exists n, $n \neq m$, such that mRn. P-4 holds as follows. Suppose G is a nonempty subset of N such that for every $n \in N$, $I(n) \subset G$ implies $n \in G$. We shall show that $G = N$. Assume that $G \neq N$. Then $N - G \neq \varnothing$. Now every nonempty subset of N has a minimal element with respect to R, since every nonempty subset of N contains a smallest positive integer. Hence $N - G$ contains a positive integer b, such that there exists no $m \in N - G$ where mRb. If $b = 1$, then $I(b) = \varnothing$ and $I(b) \subset G$. By hypothesis, $b \in G$, leading to a contradiction. If $b \neq 1$, then $I(b) = \{b - 1\}$. Since $(b-1)Rb$ and b is a minimal element of $N - G$ with respect to R, we must conclude that $b - 1 \in G$. Thus $I(b) \subset G$. By hypothesis, $b \in G$ again leading to a contradiction. Hence our initial assumption that $G \neq N$ is false, and we have P-4 true.

Examples (13), (14), (15), and (17) are sufficient to establish the independence of the postulates.

(18) $A = \{$P-1, P-2, P-3, P-4$\}$. Let N be the set W discussed at the beginning of this paper and R the relation $<$ defined there.

As we pointed out earlier, it is impossible to find a case in which P-1, P-4 are true and P-2, P-3 false, simply because P-1, P-4 true and P-3 false imply that P-2 is true. This follows immediately from Theorem (16), as we now show.

(19) THEOREM. *Let N be a set, simply ordered with respect to $<$, and let N have a last element, z. Furthermore, if G is any nonempty subset of N such that for every $x \in N$, $I(x) \subset G$ implies $x \in G$, then $G = N$. Then N has a first element.*

Proof. Suppose that N has no first element. By Theorem 16, N is empty, contradicting the hypothesis that N contains z. Hence N must have a first element.

Theorem (19) proves that the postulates for a seminatural system are not completely independent. We have, however, discovered in Theorem (19) the only dependence that exists among the postulates and their contradictions.

The author is grateful to Professor Harriet F. Montague for her assistance in the preparation of this paper.

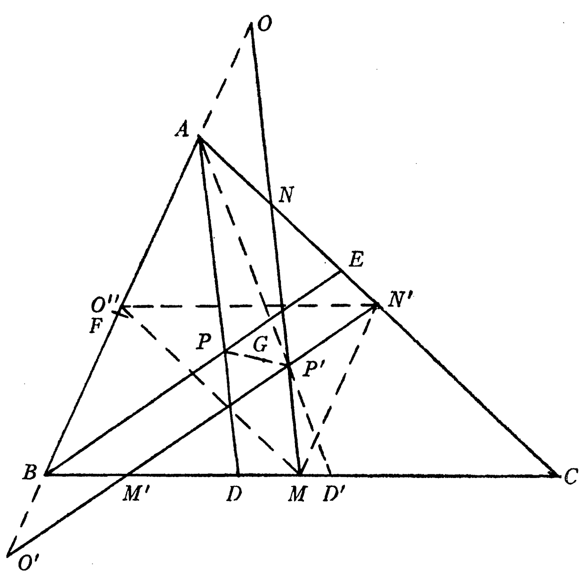
References

1. E. H. Moore, Introduction to a form of general analysis, Yale University Press, New Haven, 1910, p. 82.
2. S. T. Stern, The seminatural numbers, Amer. Math. Monthly, 73 (1966) 598–603.
3. R. L. Wilder, The foundations of mathematics, Wiley, New York, 1952, pp. 115–117.

HOMOLOGOUS POINT IN THE MEDIAL TRIANGLE

D. MOODY BAILEY, Princeton, West Virginia

M, N', O'' are the midpoints of sides BC, CA, AB of triangle ABC. Segments MN', $N'O''$, $O''M$ are parallel to sides AB, BC, CA of the given triangle and are equal to one half these respective sides. Triangle ABC and medial triangle $MN'O''$ are in perspective at their common centroid G, and this centroid is the center of similitude of these two directly similar triangles.



Proof. Suppose that N has no first element. By Theorem 16, N is empty, contradicting the hypothesis that N contains z. Hence N must have a first element.

Theorem (19) proves that the postulates for a seminatural system are not completely independent. We have, however, discovered in Theorem (19) the only dependence that exists among the postulates and their contradictions.

The author is grateful to Professor Harriet F. Montague for her assistance in the preparation of this paper.

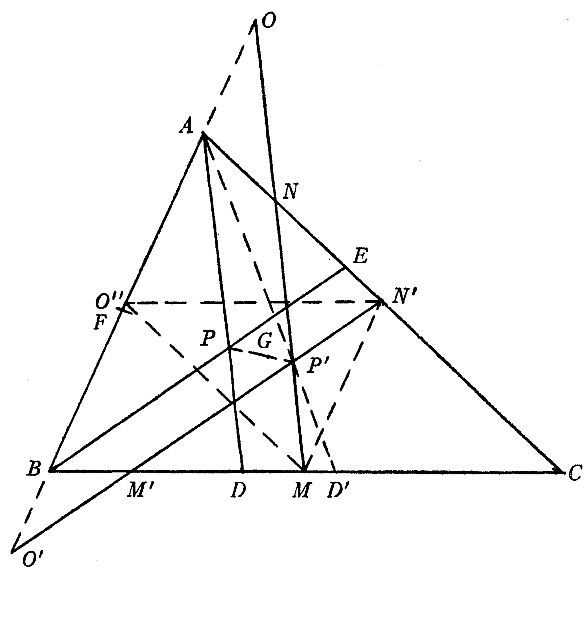
References

1. E. H. Moore, Introduction to a form of general analysis, Yale University Press, New Haven, 1910, p. 82.
2. S. T. Stern, The seminatural numbers, Amer. Math. Monthly, 73 (1966) 598–603.
3. R. L. Wilder, The foundations of mathematics, Wiley, New York, 1952, pp. 115–117.

HOMOLOGOUS POINT IN THE MEDIAL TRIANGLE

D. MOODY BAILEY, Princeton, West Virginia

M, N', O'' are the midpoints of sides BC, CA, AB of triangle ABC. Segments MN', $N'O''$, $O''M$ are parallel to sides AB, BC, CA of the given triangle and are equal to one half these respective sides. Triangle ABC and medial triangle $MN'O''$ are in perspective at their common centroid G, and this centroid is the center of similitude of these two directly similar triangles.



Let P be any point in the plane of triangle ABC, with DEF its cevian triangle. Allow P' to be a point having a position in medial triangle $MN'O''$ homologous to the position of point P in triangle ABC. Rays MP', $N'P'$, $O''P'$ will then be parallel to respective rays AP, BP, CP and segment PP' will pass through the center of similitude G, with $PG=2GP'$.

Extend MP' to meet CA and AB at points N and O respectively and let a, b, c represent sides BC, CA, AB of the given triangle. Since AD and MN are parallel, we may write

$$\frac{AN}{NC} = \frac{DM}{MC} = \frac{DC - a/2}{a/2} = \frac{1 - BD/DC}{1 + BD/DC}\,.$$

This latter value for AN/NC is obtained by replacing DC by the equivalent quantity $a/(BD/DC+1)$. Ratios BM/MC and CN/NA associated with line MNO are now known and ratio AO/OB may be found through the use of the equation of Menelaus: $(BM/MC)(CN/NA)(AO/OB) = -1$. When this is done the ratios determined by line MNO are found to be

(1) $$\frac{BM}{MC} = 1,$$

(2) $$\frac{CN}{NA} = \frac{1 + BD/DC}{1 - BD/DC},$$

(3) $$\frac{AO}{OB} = -\frac{1 - BD/DC}{1 + BD/DC}\,.$$

Allow $N'P'$ to be extended to meet sides AB and BC at respective points O' and M'. BE and $M'N'$ are parallel so that

$$\frac{BM'}{M'C} = \frac{EN'}{N'C} = \frac{EC - b/2}{b/2} = \frac{1 - AE/EC}{1 + AE/EC},$$

when EC is replaced by the quantity $b/(AE/EC+1)$. For the theorem of Menelaus to be satisfied, the ratio values for line $M'N'O'$ must be

(4) $$\frac{BM'}{M'C} = \frac{1 - AE/EC}{1 + AE/EC},$$

(5) $$\frac{CN'}{N'A} = 1,$$

(6) $$\frac{AO'}{O'B} = -\frac{1 + AE/EC}{1 - AE/EC}\,.$$

Construct ray AD' through P', the point of intersection of lines MNO and $M'N'O'$. It is then known [1] that

$$\frac{BD'}{D'C} = -\frac{BO/OA - BO'/O'A}{CN/NA - CN'/N'A}\,.$$

By substituting from (3), (6), (2), (5) in the right member of this equation, it is found that

$$\frac{BD'}{D'C}=\frac{BD/DC+AE/EC}{BD/DC+(BD/DC)(AE/EC)}.$$

Ratio AE/EC found in the numerator of this expression may be replaced by $(AF/FB)(BD/DC)$, since Ceva's equation $(BD/DC)(CE/EA)(AF/FB)=1$ shows that the two quantities are equal. BD/DC may then be removed as a factor of both numerator and denominator of the expression for $BD'/D'C$ and we have

$$\frac{BD'}{D'C}=\frac{AF/FB+1}{AE/EC+1}.$$

Rays BP' and CP' may now be constructed to meet sides CA and AB at points E' and F'. $CE'/E'A$ and $AF'/F'B$ are then determined through the use of the equations [1]

$$\frac{CE'}{E'A}=-\frac{CM/MB-CM'/M'B}{AO/OB-AO'/O'B} \quad\text{and}\quad \frac{AF'}{F'B}=-\frac{AN/NC-AN'/N'C}{BM/MC-BM'/M'C}.$$

Let points M, N', O'' be replaced by points A', B', C' so that $A'B'C'$ represents the medial triangle of triangle ABC. This result then follows:

THEOREM. *P and P' are two points in the plane of triangle ABC, with DEF and $D'E'F'$ their respective cevian triangles. If P' has a position in medial triangle $A'B'C'$ homologous to the position of point P in triangle ABC, then*

$$\frac{BD'}{D'C}=\frac{AF/FB+1}{AE/EC+1},\qquad \frac{CE'}{E'A}=\frac{BD/DC+1}{BF/FA+1},\qquad \frac{AF'}{F'B}=\frac{CE/EA+1}{CD/DB+1}.$$

As an example of the use of this theorem, let P be the symmedian point of triangle ABC so that $BD/DC=c^2/b^2$, $CE/EA=a^2/c^2$, $AF/FB=b^2/a^2$. Point P' then has ratio values

$$\frac{BD'}{D'C}=\frac{a^2+b^2}{a^2+c^2},\qquad \frac{CE'}{E'A}=\frac{b^2+c^2}{a^2+b^2},\qquad \frac{AF'}{F'B}=\frac{a^2+c^2}{b^2+c^2}$$

and P' is the symmedian point of medial triangle $A'B'C'$.

Again, allow point P to be the circumcenter of triangle ABC so that

$$\frac{BD}{DC}=\frac{c^2}{b^2}\left(\frac{a^2+b^2-c^2}{a^2+c^2-b^2}\right),\qquad \frac{CE}{EA}=\frac{a^2}{c^2}\left(\frac{b^2+c^2-a^2}{a^2+b^2-c^2}\right),$$

$$\frac{AF}{FB}=\frac{b^2}{a^2}\left(\frac{a^2+c^2-b^2}{b^2+c^2-a^2}\right).$$

P', the circumcenter of medial triangle $A'B'C'$ or nine point center of triangle ABC, will then have the ratio values

$$\frac{BD'}{D'C}=\frac{b^2(a^2+c^2-b^2)+a^2(b^2+c^2-a^2)}{c^2(a^2+b^2-c^2)+a^2(b^2+c^2-a^2)},$$

$$\frac{CE'}{E'A}=\frac{c^2(a^2+b^2-c^2)+b^2(a^2+c^2-b^2)}{a^2(b^2+c^2-a^2)+b^2(a^2+c^2-b^2)},$$

$$\frac{AF'}{F'B}=\frac{a^2(b^2+c^2-a^2)+c^2(a^2+b^2-c^2)}{b^2(a^2+c^2-b^2)+c^2(a^2+b^2-c^2)}.$$

The reader will find it worthwhile to allow P to be the incenter, either of the Brocard points, etc., of triangle ABC. The ratio values for P', the corresponding point in medial triangle $A'B'C'$, may then be calculated.

In using the preceding theorem it must be remembered that the segments involved are to be treated as directed quantities. Ratio BD/DC is considered positive when D lies between B and C, or negative when D lies on BC extended. Similar comments apply to the other ratios involved in the result given.

Reference

1. D. M. Bailey, Point of intersection of triangle transversals, this MAGAZINE, 37 (1964) 331–3.

PROOF OF THE IMPOSSIBILITY OF TRISECTING AN ANGLE WITH EUCLIDEAN TOOLS

TZER-LIN CHEN, Taipei, Taiwan

Let A be the known angle of a right triangle ABC with $AB=1$. Then BC may be taken equal to $\tan A$ and $AC=\sqrt{(1+\tan^2 A)}$. Also

$$\sin A=\frac{\tan A}{(1+\tan^2 A)^{1/2}}=\cos(90^\circ-A)$$

and

$$\tan A=\tan\left(\frac{2A}{3}+\frac{A}{3}\right)=\frac{3\tan\frac{A}{3}-\tan^3\frac{A}{3}}{1-3\tan^2\frac{A}{3}},$$

or

$$\tan^3\frac{A}{3}-3\tan A\tan^2\frac{A}{3}-3\tan\frac{A}{3}+\tan A=0. \tag{1}$$

Substituting $\tan A/3=x+\tan A$ in (1) and simplifying we obtain

$$x^3+(-3\tan^2 A-3)x+(-2\tan^3 A-2\tan A)=0. \tag{2}$$

$$\frac{BD'}{D'C} = \frac{b^2(a^2+c^2-b^2)+a^2(b^2+c^2-a^2)}{c^2(a^2+b^2-c^2)+a^2(b^2+c^2-a^2)},$$

$$\frac{CE'}{E'A} = \frac{c^2(a^2+b^2-c^2)+b^2(a^2+c^2-b^2)}{a^2(b^2+c^2-a^2)+b^2(a^2+c^2-b^2)},$$

$$\frac{AF'}{F'B} = \frac{a^2(b^2+c^2-a^2)+c^2(a^2+b^2-c^2)}{b^2(a^2+c^2-b^2)+c^2(a^2+b^2-c^2)}.$$

The reader will find it worthwhile to allow P to be the incenter, either of the Brocard points, etc., of triangle ABC. The ratio values for P', the corresponding point in medial triangle $A'B'C'$, may then be calculated.

In using the preceding theorem it must be remembered that the segments involved are to be treated as directed quantities. Ratio BD/DC is considered positive when D lies between B and C, or negative when D lies on BC extended. Similar comments apply to the other ratios involved in the result given.

Reference

1. D. M. Bailey, Point of intersection of triangle transversals, this MAGAZINE, 37 (1964) 331–3.

PROOF OF THE IMPOSSIBILITY OF TRISECTING AN ANGLE WITH EUCLIDEAN TOOLS

TZER-LIN CHEN, Taipei, Taiwan

Let A be the known angle of a right triangle ABC with $AB=1$. Then BC may be taken equal to tan A and $AC=\sqrt{(1+\tan^2 A)}$. Also

$$\sin A = \frac{\tan A}{(1+\tan^2 A)^{1/2}} = \cos(90° - A)$$

and

$$\tan A = \tan\left(\frac{2A}{3}+\frac{A}{3}\right) = \frac{3\tan\frac{A}{3} - \tan^3\frac{A}{3}}{1-3\tan^2\frac{A}{3}},$$

or

$$\tan^3\frac{A}{3} - 3\tan A\tan^2\frac{A}{3} - 3\tan\frac{A}{3} + \tan A = 0. \tag{1}$$

Substituting $\tan A/3 = x+\tan A$ in (1) and simplifying we obtain

$$x^3 + (-3\tan^2 A - 3)x + (-2\tan^3 A - 2\tan A) = 0. \tag{2}$$

Set $p=-3\tan^2 A-3$ and $q=-2\tan^3 A-2\tan A$; then

$$D=\frac{q^2}{4}+\frac{p^3}{27}=\frac{4(\tan^3 A+\tan A)^2}{4}+\frac{-27(\tan^2 A+1)^3}{27}$$
$$=-(\tan^2 A+1)^2<0.$$

Therefore the discriminant of the cubic equation (2) is negative and the three roots are real and unequal. Using De Moivre's theorem to solve equation (2), we get

$$r=\left(\frac{-p^3}{27}\right)^{1/2}=(\tan^2 A+1)^{3/2}$$

and

$$\cos\theta=\left(\frac{-q}{2}\right)\Big/\left(\frac{-p^3}{27}\right)^{1/2}=\frac{\tan A}{(\tan^2 A+1)^{1/2}}=\cos(90^\circ-A),$$

or $\theta=90^\circ-A$. Hence

$$x_1=2r^{1/3}\cos\frac{\theta}{3},\qquad x_2=2r^{1/3}\cos\frac{\theta+360^\circ}{3},\qquad x_3=2r^{1/3}\cos\frac{\theta+720^\circ}{3}$$

and

$$x_1=2(\tan^2 A+1)^{1/2}\cos\left(30^\circ-\frac{A}{3}\right),$$
$$x_2=2(\tan^2 A+1)^{1/2}\cos\left(150^\circ-\frac{A}{3}\right),$$
$$x_3=-2(\tan^2 A+1)^{1/2}\sin\frac{A}{3}\cdot$$

Therefore,

$$\tan\frac{A_1}{3}=\tan A+2(\tan^2 A+1)^{1/2}\cos\left(30^\circ-\frac{A}{3}\right)\tag{3}$$

$$\tan\frac{A_2}{3}=\tan A+2(\tan^2 A+1)^{1/2}\cos\left(150^\circ-\frac{A}{3}\right)\tag{4}$$

$$\tan\frac{A_3}{3}=\tan A-2(\tan^2 A+1)^{1/2}\sin\frac{A}{3}\cdot\tag{5}$$

If we let $A=45^\circ$, then $A/3=15^\circ$, $\tan 45^\circ=1$, $\sin 45^\circ=1/2^{1/2}$, $\tan 15^\circ=2-3^{1/2}$, $\sin 15^\circ=(6^{1/2}-2^{1/2})4$, $\cos 15^\circ=(6^{1/2}+2^{1/2})/4$. Substituting these values in both members of (3), (4), and (5), we find that only (5) is satisfied. Consequently (5) is the exact solution of equation (1).

$$\tan\frac{A}{3} = \tan A - 2(\tan^2 A + 1)^{1/2}\sin\frac{A}{3}; \tag{5}$$

therefore

$$\sin\frac{A}{3} = \frac{\tan A - \tan\frac{A}{3}}{2(\tan^2 A + 1)^{1/2}} \tag{6}$$

and also

$$\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3}. \tag{7}$$

If (6) is substituted into (7), we obtain

$$\frac{\tan A}{(\tan^2 A + 1)^{1/2}} = \frac{3\tan A - 3\tan\frac{A}{3}}{2(\tan^2 A + 1)^{1/2}} - \frac{4\left(\tan A - \tan\frac{A}{3}\right)^3}{8(\tan^2 A + 1)^{3/2}}$$

which when simplified becomes

$$\tan^3\frac{A}{3} - 3\tan A\left(\tan^2\frac{A}{3}\right) - 3\tan\frac{A}{3} + \tan A = 0.$$

Since equation (7) can be transformed into equation (1), equation (5) is the exact solution of equation (1).

In equation (5), $\tan A$ is a known value but $\sin A/3$ is an unknown value, therefore we cannot determine the value of $\tan A/3$. Suppose, in Figure 1 $\angle BAD = 1/3\angle A$; then $\tan A/3 = BD/AB = BD$. If we could trisect an angle with Euclidean tools, we could determine the value of $\tan A/3$, but because this is impossible we cannot trisect an angle.

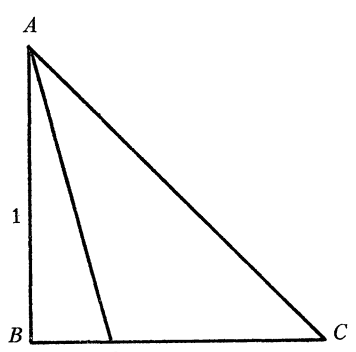


FIG. 1.

BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114.

Groups and their Graphs. By Israel Grossman and Wilhelm Magnus. Random House and L. W. Singer Co., New York, 1964. vii+195 pp. $1.95 (paper).

Bringing the flavor of modern abstract algebra to beginning undergraduate students is the primary concern of this brief, but well conceived, monograph. Implicit in its goals is the establishment of relations between abstract mathematical structures and numbers. For the sake of concreteness the authors adopt the notion of a *group* as the paradigm *par excellence* for algebraic structures.

In order to make good use of the student's natural intuition the writers employ the usual four axioms for a group involving left and right identities together with left and right inverses. Thus, at a slight cost of conciseness and possibly elegance, they may well have purchased better understanding of group theory for a wider audience than mathematics majors; i.e., they invoke here, as well as elsewhere, the useful pedagogical principle of the conservation of logic and psychology.

The book is replete with well chosen elementary examples of groups from both geometrical and arithmetical contexts. Special emphasis is given to finite groups, i.e., finite sets which satisfy the group axioms. No attempt is made to relax any of the axioms in order to study more general algebra structures. Graphs are introduced as pictorial models for various kinds of groups, thereby tapping the topological intuition of the student.

Each of the authors' examples opens up an easily traversed foot-path leading to the highway of graduate mathematics, e.g., a *word* on generators leads to a study of very simple, but meaningful, finitely presented groups. If one were to pursue these groups in more advanced works one would soon come to the problem of groups with solvable word-problems, a professional favorite of the second-named author; and thence to the problem of groups with recursively unsolvable word-problems. Similarly, the authors' clear treatment of residue systems nearly begs the reader to define the Gaussian sum, the Ramanujan sum, and the Kloosterman sum of number theory.

Among the wide range of topics treated are: (1) the symmetric and alternating groups so useful in establishing, e.g., Abel's result on the unsolvability, by radicals, of the general quintic equation (solvable groups, *as such*, are not studied), (2) the quaternion group, (3) simple groups, (4) one of Sylow's theorems (the "converse" of Lagrange's theorem), and (5) introductory knot theory. They even state correctly on page 149, contrary to an assertion in Birkhoff and MacLane (Macmillan Company, 1953, and later edition, page 455) that the alternating group on 5 letters, A_5, is " . . . the smallest non-Abelian group with no normal proper subgroups."

Solutions are provided for each of the 65 interesting exercises listed in this valuable paperback book. The reviewer's only criticisms are: (1) on page 14, most likely Galois introduced the word "group" *before* 1832; (2) on page 25, exercise 4 should probably be stated as a uniqueness result in addition to an existence result; and (3) on page 120, for "Galois died at the age of 21 . . . " read "Galois died at the age of 20 . . . ."

A. A. MULLIN, University of California

Graphs and their Uses. By Oystein Ore. Random House and L. W. Singer Co., New York, 1963. vii+131 pp. $1.95 (paper).

In recent years enthusiasm for the study of combinatorial graphs (*not* the kind wherein one "plots y against x") has been waxing, not only in its theoretical departments but in its practical technological applications, too. Probably the first significant paper on graph theory (roughly, the study of topological properties of "points" joined by "arcs" without regard to metrical considerations) was published by the 29-year-old L. Euler during his first stay at St. Petersburg. It dealt with abstractions from the famous problem of the seven bridges of Königsberg in East Prussia, *viz.*, could one take a stroll and return home so as to cross each of the seven bridges once and only once? Initially graph theory was not taken seriously by many mathematicians. It was considered as a motley of recreational problems. However, increased popularity came to finite graph-theory through the theoretical efforts of W. R. Hamilton (who studied the dual of Euler's problem, i.e., traversing each vertex of a graph once and only once), and the practical investigations of the physicist G. R. Kirchhoff (who gave graph-theoretical accounts of electric circuits) and of many organic chemists analyzing molecular diagrams. At present graph theory is an important analytical tool for the study of, e.g., sequential switching circuits, inventories, abstract games, optimal flow in complex chemical networks, the optimal routing of commodities, and optimal waiting-times for queues such as occur in the check-out lines of supermarkets.

This carefully executed monograph, replete with theory and proof, examples, and applications of finite combinatorial graph-theory is intended for beginning undergraduate students. The author commences with an abstract, but quite readable introduction to graphs; e.g., he defines null graph, universal graphs, and isomorphic graphs. Ideas and theorems are formulated smoothly and at a reasonable pace, although the reviewer was somewhat distressed to find the statement (without proof!) of the Jordan Curve Theorem as early as page 15. There are excellent, elementary discussions of the graphs of Euler and of Hamilton mentioned earlier in this review. Further, there are useful treatments of the notion of "tree" (so valuable for studying linear electric-circuit theory), directed graphs (convenient for investigating systems including unilateral circuit elements such as diodes and, also, other one-way traffic), and planar graphs. No explicit mention is given to either graphs with "weighted" edges or "weighted" vertices, to colored graphs, or to infinite graphs. The book ends with a brief discussion of the famous four-color conjecture of Möbius, Guthrie, and Cayley, *viz.*, are four colors sufficient to color the countries of any planar map so that countries with a common boundary have different colors? This simply stated problem is, for combinatorial topologists, similar in difficulty to Fermat's Last "Theorem" for most likely Galois introduced the word "group" *before* 1832; (2) on page 25, exercise 4 should probably be stated as a uniqueness result in addition to an existence result; and (3) on page 120, for "Galois died at the age of 21 . . . " read "Galois died at the age of 20 . . . ."

A. A. MULLIN, University of California

Graphs and their Uses. By Oystein Ore. Random House and L. W. Singer Co., New York, 1963. vii+131 pp. $1.95 (paper).

In recent years enthusiasm for the study of combinatorial graphs (*not* the kind wherein one "plots y against x") has been waxing, not only in its theoretical departments but in its practical technological applications, too. Probably the first significant paper on graph theory (roughly, the study of topological properties of "points" joined by "arcs" without regard to metrical considerations) was published by the 29-year-old L. Euler during his first stay at St. Petersburg. It dealt with abstractions from the famous problem of the seven bridges of Königsberg in East Prussia, *viz.*, could one take a stroll and return home so as to cross each of the seven bridges once and only once? Initially graph theory was not taken seriously by many mathematicians. It was considered as a motley of recreational problems. However, increased popularity came to finite graph-theory through the theoretical efforts of W. R. Hamilton (who studied the dual of Euler's problem, i.e., traversing each vertex of a graph once and only once), and the practical investigations of the physicist G. R. Kirchhoff (who gave graph-theoretical accounts of electric circuits) and of many organic chemists analyzing molecular diagrams. At present graph theory is an important analytical tool for the study of, e.g., sequential switching circuits, inventories, abstract games, optimal flow in complex chemical networks, the optimal routing of commodities, and optimal waiting-times for queues such as occur in the check-out lines of supermarkets.

This carefully executed monograph, replete with theory and proof, examples, and applications of finite combinatorial graph-theory is intended for beginning undergraduate students. The author commences with an abstract, but quite readable introduction to graphs; e.g., he defines null graph, universal graphs, and isomorphic graphs. Ideas and theorems are formulated smoothly and at a reasonable pace, although the reviewer was somewhat distressed to find the statement (without proof!) of the Jordan Curve Theorem as early as page 15. There are excellent, elementary discussions of the graphs of Euler and of Hamilton mentioned earlier in this review. Further, there are useful treatments of the notion of "tree" (so valuable for studying linear electric-circuit theory), directed graphs (convenient for investigating systems including unilateral circuit elements such as diodes and, also, other one-way traffic), and planar graphs. No explicit mention is given to either graphs with "weighted" edges or "weighted" vertices, to colored graphs, or to infinite graphs. The book ends with a brief discussion of the famous four-color conjecture of Möbius, Guthrie, and Cayley, *viz.*, are four colors sufficient to color the countries of any planar map so that countries with a common boundary have different colors? This simply stated problem is, for combinatorial topologists, similar in difficulty to Fermat's Last "Theorem" for

algebraists, and the Riemann Hypothesis for analysts. It is shown that five colors are sufficient for any planar map. Configurations in 3-space, each of which requires infinitely many colors, are not treated.

This thoroughly readable paperback book is highly recommended for advanced lower-division students and their teachers. Each of the 50 exercises and its solution shows good planning. *Erratum.* On page 129, in the Glossary under the term "dodecahedron" read "twelve faces" for "twenty faces."

A. A. MULLIN, University of California

Number Systems: A Modern Introduction. By Mervin L. Keedy. Addison-Wesley, Reading, Massachusetts, 1965. vi+226 pp. $5.50.

The material in this text is intended as a one semester course for liberal arts students with only two years of high school mathematics and for prospective elementary school teachers. The topics have been selected from the author's earlier book *A Modern Introduction to Basic Mathematics* and expanded to meet the needs of the particular groups mentioned above. There are eight chapters in the book. One chapter treats numeration. Three chapters are devoted to abstract mathematical systems, sentences, sets, relations, logic, and mathematical reasoning. The remaining four chapters deal with natural numbers, integers, rational and real numbers.

The treatment of the topics in the various chapters is uneven and rather sketchy. An attempt has been made to pass from an intuitive discussion of number systems to some level of modern rigor. The transition is not always smooth. There are two sets of exercises in the book. One set of review practice exercises is only incidentally related to the subject matter of the sections. The other set of exercises is intended to test the proficiency of the student in mastering the material of the chapter. The chapters on natural numbers, the numbers of arithmetic, and the integers are clear, well illustrated, and the best in the book. The student with no background other than that stated by the author will have some difficulty with the material on abstract mathematical systems and logic. Some of the definitions encountered here seem to be ambiguously stated.

The last chapter, of approximately eighteen pages, discusses the rational numbers and real numbers. Perhaps such a brief treatment is appropriate for elementary teachers. However, the title of the book leads one to expect a somewhat more detailed discussion. This is especially annoying since elementary teachers and students in general are weakest when questioned on the rational and real numbers. The structural approach to elementary number systems developed in this book will prove profitable reading for both teachers and students whose previous experience with number systems has been mainly on the manipulative level.

S. J. BEZUSZKA, S.J., Boston College

BRIEF MENTION

Most Significant New Books on Mathematics/1964. Hans M. Zell, Editor. Robert Maxwell Documentation and Supply Centre, Waynflete Building, 1-8, St. Clements, Oxford, England.

algebraists, and the Riemann Hypothesis for analysts. It is shown that five colors are sufficient for any planar map. Configurations in 3-space, each of which requires infinitely many colors, are not treated.

This thoroughly readable paperback book is highly recommended for advanced lower-division students and their teachers. Each of the 50 exercises and its solution shows good planning. *Erratum.* On page 129, in the Glossary under the term "dodecahedron" read "twelve faces" for "twenty faces."

A. A. MULLIN, University of California

Number Systems: A Modern Introduction. By Mervin L. Keedy. Addison-Wesley, Reading, Massachusetts, 1965. vi+226 pp. $5.50.

The material in this text is intended as a one semester course for liberal arts students with only two years of high school mathematics and for prospective elementary school teachers. The topics have been selected from the author's earlier book *A Modern Introduction to Basic Mathematics* and expanded to meet the needs of the particular groups mentioned above. There are eight chapters in the book. One chapter treats numeration. Three chapters are devoted to abstract mathematical systems, sentences, sets, relations, logic, and mathematical reasoning. The remaining four chapters deal with natural numbers, integers, rational and real numbers.

The treatment of the topics in the various chapters is uneven and rather sketchy. An attempt has been made to pass from an intuitive discussion of number systems to some level of modern rigor. The transition is not always smooth. There are two sets of exercises in the book. One set of review practice exercises is only incidentally related to the subject matter of the sections. The other set of exercises is intended to test the proficiency of the student in mastering the material of the chapter. The chapters on natural numbers, the numbers of arithmetic, and the integers are clear, well illustrated, and the best in the book. The student with no background other than that stated by the author will have some difficulty with the material on abstract mathematical systems and logic. Some of the definitions encountered here seem to be ambiguously stated.

The last chapter, of approximately eighteen pages, discusses the rational numbers and real numbers. Perhaps such a brief treatment is appropriate for elementary teachers. However, the title of the book leads one to expect a somewhat more detailed discussion. This is especially annoying since elementary teachers and students in general are weakest when questioned on the rational and real numbers. The structural approach to elementary number systems developed in this book will prove profitable reading for both teachers and students whose previous experience with number systems has been mainly on the manipulative level.

S. J. BEZUSZKA, S.J., Boston College

BRIEF MENTION

Most Significant New Books on Mathematics/1964. Hans M. Zell, Editor. Robert Maxwell Documentation and Supply Centre, Waynflete Building, 1-8, St. Clements, Oxford, England.

algebraists, and the Riemann Hypothesis for analysts. It is shown that five colors are sufficient for any planar map. Configurations in 3-space, each of which requires infinitely many colors, are not treated.

This thoroughly readable paperback book is highly recommended for advanced lower-division students and their teachers. Each of the 50 exercises and its solution shows good planning. *Erratum.* On page 129, in the Glossary under the term "dodecahedron" read "twelve faces" for "twenty faces."

A. A. MULLIN, University of California

Number Systems: A Modern Introduction. By Mervin L. Keedy. Addison-Wesley, Reading, Massachusetts, 1965. vi+226 pp. $5.50.

The material in this text is intended as a one semester course for liberal arts students with only two years of high school mathematics and for prospective elementary school teachers. The topics have been selected from the author's earlier book *A Modern Introduction to Basic Mathematics* and expanded to meet the needs of the particular groups mentioned above. There are eight chapters in the book. One chapter treats numeration. Three chapters are devoted to abstract mathematical systems, sentences, sets, relations, logic, and mathematical reasoning. The remaining four chapters deal with natural numbers, integers, rational and real numbers.

The treatment of the topics in the various chapters is uneven and rather sketchy. An attempt has been made to pass from an intuitive discussion of number systems to some level of modern rigor. The transition is not always smooth. There are two sets of exercises in the book. One set of review practice exercises is only incidentally related to the subject matter of the sections. The other set of exercises is intended to test the proficiency of the student in mastering the material of the chapter. The chapters on natural numbers, the numbers of arithmetic, and the integers are clear, well illustrated, and the best in the book. The student with no background other than that stated by the author will have some difficulty with the material on abstract mathematical systems and logic. Some of the definitions encountered here seem to be ambiguously stated.

The last chapter, of approximately eighteen pages, discusses the rational numbers and real numbers. Perhaps such a brief treatment is appropriate for elementary teachers. However, the title of the book leads one to expect a somewhat more detailed discussion. This is especially annoying since elementary teachers and students in general are weakest when questioned on the rational and real numbers. The structural approach to elementary number systems developed in this book will prove profitable reading for both teachers and students whose previous experience with number systems has been mainly on the manipulative level.

S. J. BEZUSZKA, S.J., Boston College

BRIEF MENTION

Most Significant New Books on Mathematics/1964. Hans M. Zell, Editor. Robert Maxwell Documentation and Supply Centre, Waynflete Building, 1-8, St. Clements, Oxford, England.

Free copies of this list of 114 titles may be obtained from the above address (while the supply lasts). Included are short descriptive annotations or review extracts.

Guidebook to Departments in the Mathematical Sciences in the United States and Canada. Raoul Hailpern, Editor. The Mathematical Association of America, SUNY at Buffalo, Buffalo, New York 14214, 1965. 64 pp. $.50 (paper).

This Guidebook provides in summary form information about the location, size, staff, library facilities, course offerings, and special features of departments in mathematical sciences in four year colleges and universities.

Algorithms and Automatic Computing Machines. By B. A. Trakhtenbrot. Heath, Boston, 1963. 101 pp. $1.70 (paper).

In the *Topics in Mathematics Series;* translated and adapted from the second Russian edition (1960). An eminently readable account of algorithms and Turing machines.

Teaching Elementary School Mathematics for Understanding, 2nd ed. By John L. Marks, C. Richard Purdy, and Lucien B. Kinney. McGraw-Hill, New York, 1965. 500 pp. $7.95.

A revision of the 1958 edition; exceptionally attractive format.

Algebra and Trigonometry. By A. P. Hillman and G. L. Alexanderson. Allyn and Bacon, Boston, 1963. xiv+514 pp. $7.95.

A treatment of algebra and trigonometry integrated with vectors, complex numbers, and elementary functions.

Analytic Geometry, 2nd ed. By W. K. Morrill. International Textbook, Scranton, Pa., 1964. x+435 pp. $5.75.

Appropriate for a one or two semester course. This revision contains a more extensive discussion of curve sketching and parametric equations; an introduction to matrix theory is included in an appendix.

Breakthroughs in Mathematics. By Peter Wolff. The New American Library, New York, 1963. 285 pp. $.75 (paper).

This Signet Science Library book includes excerpts from the works of Euclid, Lobachevski, Decartes, Archimedes, Dedekind, Russell, Euler, Laplace, and Boole, together with numerous commentaries.

Principles of Modern Mathematics, Book I. By William E. Hartnett. Harper and Row, New York, 1963. xiii+416 pp. $7.75.

A formal but fresh treatment of the real number system, functions, algebraic structures, sequences, limits, and other topics from classical analysis.

Programming the IBM 1620, 2nd ed. By Clarence B. Germain. Prentice-Hall, Englewood Cliffs, N. J., 1965. 191 pp. $4.95 (paper).

Includes discussion of SPS, various versions of Fortran, disk storage, and all features available on the Model 1 or 2.

Algebra: A Modern Introduction. By J. L. Kelley. D. Van Nostrand, Princeton, N. J., 1965. viii+335 pp. $6.75.

Intended for a standard precalculus course in algebra; suitable for students of physical sciences, mathematics, or the social sciences. Vector geometry and linear algebra are included.

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in india ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.

PROPOSALS

628. *Proposed by B. Suer and Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the alphametic,

$$\text{COS}^2 + \text{SIN}^2 = \text{UNO}^2$$

in the decimal system.

629. *Proposed by C. Stanley Ogilvy and Stephen Barr, Hamilton College, New York.*

Rectangle $OPQR$ is initially placed so that OP lies along the positive x-axis and OR lies along the positive y-axis. If the rectangle is rotated through 90° in such a way that O slides along the x-axis and R slides along the y-axis, what is the locus of Q?

630. *Proposed by C. J. Mozzochi, University of Connecticut.*

Prove: If f is real valued, strictly monotone increasing, and defined almost everywhere on $[a, b]$, then f^{-1} is continuous everywhere on its domain.

631. *Proposed by J. A. H. Hunter, Toronto, Ontario, Canada.*

My first and last
add up to four.
My middle two
to just eight more.

My second from
my fifth makes three.
I'm square! What must
my whole six be?

632. *Proposed by Erwin Just, Bronx Community College.*

Prove that

$$1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1-(n\pi)^2} = \frac{1}{\sin 1}\,.$$

633. *Proposed by Dov Avishalom, Hebrew University, Jerusalem.*

Prove that the number

$$(1965^{1966} + 1968^{1967})^{0.5}$$

is irrational.

634. *Proposed by R. S. Luthar and Stephen Wurzel, Colby College, Maine.*

If p is a prime, such that

$$p^2 \not\equiv p \pmod 3$$

show that

$$p^{2n-1} + p^{2n-3} + \cdots + p + n \equiv 0 \pmod 3.$$

Errata. The second line of A376, page 42, January, 1966, should read, "arrangements in the answer to Q313 [May, 1963] may be used in successive layers, i.e., . . . "

The name of *David M. Homa, Montreal, Canada,* was omitted as joint proposer of Problem 613, January, 1966.

SOLUTIONS

Late Solutions

Sister M. Stephanie Sloyan, Georgian Court College, Lakewood, New Jersey: **600.**

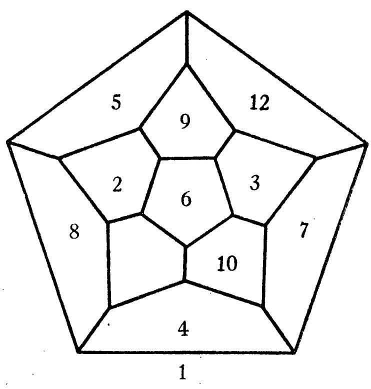
POLYHEDRAL FACES

607. [January, 1966] *Proposed by Sidney Kravitz, Picatinny Arsenal, Dover, New Jersey.*

Number the faces of a dodecahedron with the numbers 1 through 12. It is easy to place the numbers such that those on any two adjacent pentagonal faces differ by at least 2. Show that it is impossible to place the numbers such that those on any two adjacent faces differ by at least 3.

Solution by Michael Goldberg, Washington, D. C.

Draw the Schlegel diagram of the dodecahedron as shown. Label the base



face as 1. Then 2 and 3 cannot appear in the outer ring of faces. Furthermore, neither can be placed in the center since it would prevent the other from being placed in the inner ring. Hence 2 and 3 must appear in the inner ring and be separated by a face. Therefore, 4 cannot be placed in the inner ring or in the center, but can be placed only as shown. Subsequently, there is only one location for 5, 6, 7, 8, 9, 10. In the two remaining places, 11 cannot be placed, although 12 can be placed. Hence, the desired ordering is impossible.

Also solved by Lindley J. Burton, Lake Forest College, Illinois; Buster Dunsmore, University of Tennessee; Hwa S. Hahn, Pennsylvania State University; Stanley Rabinowitz, Far Rockaway, New York; C. R. J. Singleton, Petersham, Surrey, England; Alva Smufa, Philadelphia, Pennsylvania; and Gary B. Weiss, New York University School of Medicine.

THE SEMI-PERFECT NUMBER

608. [January, 1966] *Proposed by A. A. Gioia and A. M. Vaidya, Texas Technological College.*

Call a positive integer n semi-perfect if the sum of all the square free divisors of n is $2n$. Prove that 6 is the only semi-perfect number.

Solution by Stanley Rabinowitz, Far Rockaway, New York.

Suppose

$$n = (p_1)^{a_1}(p_2)^{a_2}(p_3)^{a_3} \cdots (p_k)^{a_k}(p_i < p_{i+1}).$$

The sum of the square free divisors of n is the same as the sum of the divisors of $p_1p_2p_3 \cdots p_k$ which is $(p_1+1)(p_2+1) \cdots (p_k+1)$. If n is semi-perfect, then

$$(1) \quad 2(p_1)^{a_1}(p_2)^{a_2}(p_3)^{a_3} \cdots (p_k)^{a_k} = (p_1 + 1)(p_2 + 1)(p_3 + 1) \cdots (p_k + 1).$$

It is clear that $k>1$. Hence n must be even; for if it were odd, then 2 would divide the left of (1), but 4 would divide the right. Therefore $p_1=2, p_1+1=3$, so $p_2=3$. Now p_k divides the left of (1) but not the right $(k>2)$. Hence $k=2$. Therefore we find $a_1=1$ and $a_2=1$. Therefore 6 is the only semi-perfect number.

Also solved by Arthur Bolder, Brooklyn, New York; Lindley J. Burton, Lake Forest College, Illinois; L. Carlitz, Duke University; Hwa S. Hahn, Pennsylvania State University; Sam Newman, Atlantic City, New Jersey; Bob Prielipp University of Wisconsin; C. R. J. Singleton, Petersham, Surrey, England; K. L. Yocom, South Dakota State University; and the proposers. Note. This problem appeared as problem E 1755 in the American Mathematical Monthly, Feb. 1966, page 203.

CRYPTA-EQUIVALENCE

609. [January, 1966] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the following cryptarithm in the decimal system:

$$4 \cdot NINE = 9 \cdot FOUR$$

Solution by J. D. E. Konhauser, University of Minnesota.

The products $4 \cdot E$ and $9 \cdot R$ must have the same units digit. Therefore, the only possible (E, R) combinations are (1, 6), (3, 8), (5, 0), (7, 2), and (9, 4). Since 4 and 9 are relatively prime, 9 must divide $2N+I+E$.

Case (1, 6): If $E=1$, 9 must divide $2N+I+1$, leading to the following (N, I) combinations: (4, 0), (3, 2), (7, 3), (2, 4), (5, 7), (9, 8), and (4, 9). The corresponding values for $FOUR$ are 1796, 1436, 3276, 1076, 2556, 4396, and 2196. The first two, the fourth, and the last must be rejected since $E=1$. The third is out since $N=7$. The fifth is out since 5 is repeated. The sixth is out since $N=9$.

Similar analysis applied to the remaining cases leads to the solutions given below:

Case (3, 8): $4\cdot 4743=9\cdot 2108$.
Case (5, 0): $4\cdot 6165=9\cdot 2740$.
Case (7, 2): $4\cdot 6867=9\cdot 3052$.
Case (9, 4): $4\cdot 5859=9\cdot 2604$.

Also solved by Monte Dernham, San Francisco, California; Samuel P. Hoyle, Jr., University of Virginia; Sidney Kravitz, Dover, New Jersey; C. C. Oursler, Southern Illinois University (Edwardsville); Richard Riggs, Jersey City State College, New Jersey; Jerome J. Schneider, Chicago, Illinois; and Charles W. Trigg, San Diego, California.

Partial solutions were submitted by Merrill Barneby, Wisconsin State University (La Crosse); Charles R. Berndtson, Massachusetts Institute of Technology; Lindley J. Burton, Lake Forest College, Illinois; Anton Glaser, Pennsylvania State University (Ogontz); J. A. H. Hunter, Toronto, Canada; Beatriz Margolis, University of Maryland; John W. Milsom, Slippery Rock State College, Pennsylvania; William L. Mrozek, Wyandotte, Michigan; Sam Newman, Atlantic City, New Jersey; C. R. J. Singleton, Petersham, Surrey, England; Lowell Van Tassel, San Diego City College; Gary B. Weiss, New York University, School of Medicine; Donald R. Wilder, Rochester, New York; Dale Woods, Missouri State Teachers College; and the proposer.

AN ERDÖS-MORDELL EXTENSION

610. [January, 1966] *Proposed by Leon Bankoff, Los Angeles, California.*

According to the Erdös-Mordell theorem,

$$AI + BI + CI \geqq 6r$$

where I is the incenter and r the inradius of triangle ABC, with equality when the triangle is equilateral. Show that

$$AP + BP + CP \geqq 6r$$

where P is any point within triangle ABC.

Solution by Michael Goldberg, Washington, D. C.

The minimum sum $AP+BP+CP$ occurs when P is one of the isogonic points of the triangle. Erect the equilateral triangle BCD on the base BC. Then,

$$AP + BP + CP = AD.$$

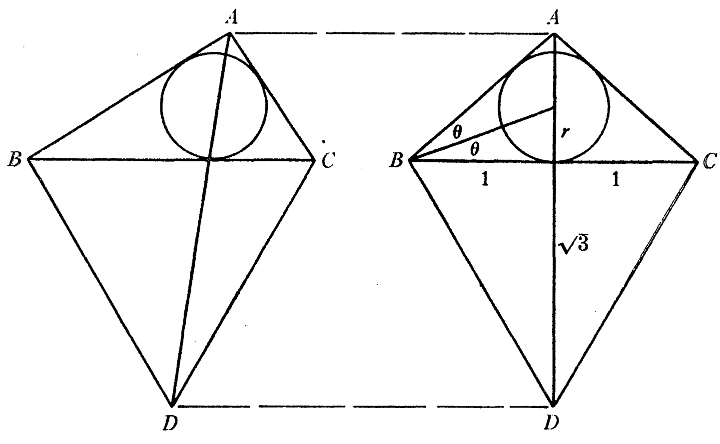
(See Advanced Euclidean Geometry, R. H. Johnson, pp. 218–220.) For a fixed base BC and a fixed height, the ratio AD/r is minimized by changing to an isosceles triangle since r is increased thereby, while AD is decreased. If $BC=2$, and $\theta=B/2$, then,

$$\begin{aligned} AD/r &= y = (\sqrt{3} + \tan 2\theta)/\tan\theta \\ &= \sqrt{3}\cot\theta + 2/(1+\tan^2\theta) \\ &= \sqrt{3}\cot\theta + 2\cos^2\theta, \\ dy/dx &= \sqrt{3}\csc^2\theta - 4\cos\theta\sin\theta = 0, \end{aligned}$$

or

$$\sqrt{3}/4 = \sin^3\theta\cos\theta.$$

Hence, the ratio AD/r is minimized when $\theta = 30°$ and ABC is equilateral. In this case, $AD/r = 6$. Hence, $AP + BP + CP \geqq 6r$ for all positions of P in the triangle.



Also solved by G. L. N. Rao, J. C. College, Jamshedpur, India and the proposer.

L. Carlitz remarked that this result is proved in L. Fejes Tóth, Lagerungen in der Ebene auf der Kugel und im Raum. (Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 65, p. 11).

J. D. E. Konhauser found the problem appearing as Problem 33 in Kazarinoff's Geometrical Inequalities.

A BOX OF TRIANGULAR NUMBERS

611. [January, 1966] *Proposed by A. Struyk, Paterson, New Jersey.*

A well-known problem is that in which a rectangular sheet of given dimensions (length L, width W) is to have cut from its corners a square (side x) so that, when the resulting figure is folded to form an open box, the box will have maximum volume.

(a) Let it be required to choose integers L and W for which x is rational. Show that proper manipulation of four suitable numbers in arithmetic sequence yields appropriate values for L, W, and x.

(b) Show that there are sheets with integral dimensions which, when cut as specified, yield maximum volume boxes whose dimensions are triangular numbers.

Solution by Michael Goldberg, Washington, D. C.

(a) Since the volume V is given by

$$V = x(L - x)(W - x) = LWx - Lx^2 - Wx^2 + x^3,$$

then

$$dV/dx = LW - 2Lx - 2Wx + 3x^2 = 0,$$

and the maximum volume is obtained when

$$x = \{(L + W) - \sqrt{(L^2 - LW + W^2)}\}/3.$$

If x is rational, then $L^2 - LW + W^2 = z^2$. Solutions for this Diophantine equation are given in History of the Theory of Numbers, L. E. Dickson, vol. 2, p. 405. They are $L = p^2 - q^2$, $W = 2pq - q^2$, $z = p^2 - pq + q^2$, for pairs of integers p, q. Then,

$$x = (L + W - z)/3 = pq - q^2.$$

For example, if p, $q = 3$, 1 then L, W, z, $x = 8$, 5, 7, 2.

(b) The dimensions of the box are

$$L - 2x = (p - q)^2,$$
$$W - 2x = q^2,$$
$$x = q(p - q).$$

Since two of the dimensions are squares, it is necessary to select those triangular numbers which are squares. They are the squares of 1, 6, 35, 204, 1189, 6930, etc. They are ascribed to Euler on page 10, of Dickson's "History," vol. 2. From these, we obtain the following sample solutions.

$L-2x$	$W-2x$	x	L	W
1	1	1	3	3
6^2	1	6	48	13
35^2	6^2	210	1645	456

Note that the derived values of x are also triangular numbers.

Also solved by C. R. J. Singleton, Petersham, Surrey, England; Charles W. Trigg, San Diego, California, and the proposer.

A TRIPLE INTEGRAL

612. [January, 1966] *Proposed by M. B. McNeil, University of Missouri at Rolla.*

The integral

$$I_1 = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{dudvdw}{1 - \cos u \cos v \cos w}$$

occurs in the study of ferromagnetism and in the study of lattice vibrations. Prove that

$$I_1 = (4\pi^3)^{-1}[\Gamma(1/4)]^4.$$

Solution by Henry T. Fettis, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio.

The above result was first obtained by G. N. Watson (Quart. J. Math, vol. 10, 1939, pp. 267–276). Watson used the substitutions

$$x = \tan \tfrac{1}{2}u$$
$$y = \tan \tfrac{1}{2}v$$
$$z = \tan \tfrac{1}{2}w$$

and then transformed to spherical coordinates. A simpler derivation consists of first performing the integration with respect to w, giving

$$I_1 = \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \frac{dudv}{\sqrt{(1 - \cos^2 u \cos^2 v)}}\,.$$

Setting

$$\cos u = \frac{1 - x^2}{1 + x^2},$$

$$\cos v = \frac{1 - y^2}{1 + y^2},$$

$$du = \frac{2dx}{1 + x^2},$$

$$dv = \frac{2dy}{1 + y^2},$$

$$I_1 = \frac{4}{\pi^2}\int_0^\infty \int_0^\infty \frac{dxdy}{\sqrt{((1 + x^2)^2(1 + y^2)^2 - (1 - x^2)^2(1 - y^2)^2)}}$$
$$= \frac{2}{\pi^2}\int_0^\infty \int_0^\infty \frac{dxdy}{\sqrt{((x^2 + y^2)(1 + x^2y^2))}}\,.$$

Changing to polar coordinates

$$I_1 = \frac{2}{\pi^2}\int_0^\infty \int_0^{\pi/2} \frac{\rho d\rho d\theta}{\sqrt{(1 + \rho^4 \sin^2 \theta \cos^2 \theta)}}$$
$$= \frac{2}{\pi^2}\int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin \theta \cos \theta)}} \int_0^\infty \frac{dt}{\sqrt{(1 + t^4)}}$$

where the substitution $t^4 = \rho^4 \sin^2 \theta \cos^2 \theta$ has been made in the second integral, since

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin \theta \cos \theta)}} = \frac{[\Gamma(\frac{1}{4})]^2}{2\sqrt{\pi}}$$

and

$$\int_0^\infty \frac{dt}{\sqrt{(1+t^4)}} = \frac{[\Gamma(\frac{1}{4})]^2}{4\sqrt{\pi}},$$

$$I_1 = \frac{1}{4\pi^3}[\Gamma(\tfrac{1}{4})]^4.$$

L. Carlitz and the proposer also referred to G. N. Watson's solution.

H. W. Gould and William Squire, West Virginia University (jointly), pointed out that

$$\left[\Gamma\left(\frac{1}{4}\right)\right]^4 = 8\pi^2 \cdot \frac{3}{1} \cdot \frac{3}{5} \cdot \frac{7}{5} \cdot \frac{7}{9} \cdot \frac{11}{9} \cdots$$

which gives another way of expressing the integral. This formula was found in E. Catalan in 1883 (Petersburg Memoirs, vol. 31, pp. 1–51). They also found a pertinent reference in Tables of Integrals by Ryshik and Gradstein, page 619, English Translation, Academic Press.

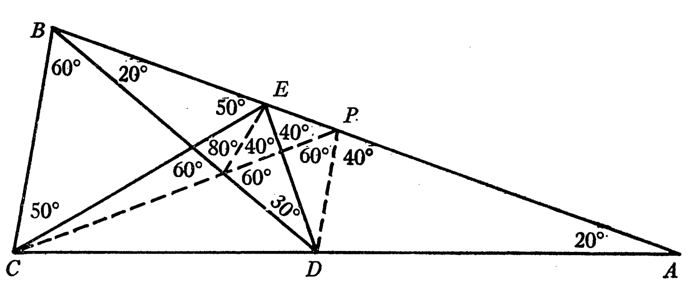
ISOSCELES TRIANGLE PROBLEM

613. [January, 1966] *Proposed by J. A. H. Hunter, Toronto, Ontario, Canada and David M. Homa, Montreal, Canada.*

In the isosceles triangle ABC, we have $\angle ABC = \angle ACB = 80°$. With E on side AB, $\angle BCE = 50°$, and with D on side AC, $\angle CBD = 60°$. By purely geometrical considerations, and without use of any trigonometrical techniques, evaluate the angle BDE.

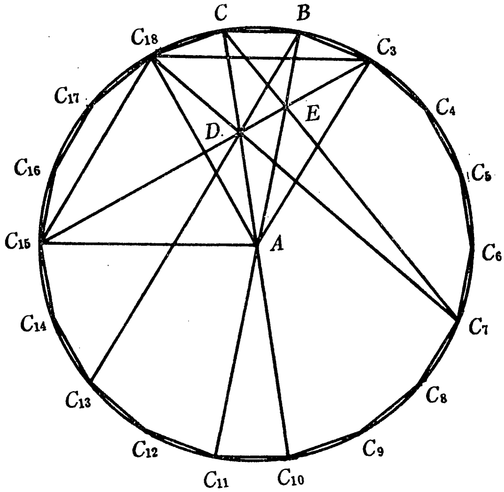
Solution by Charles W. Trigg, San Diego, California.

Method I. Draw DP parallel to BC, PC intersecting BD at M, and draw EM. By symmetry, $\angle MCB = \angle CBD = 60°$, so triangle BCM is equiangular, as is triangle MPD, with every angle equal to $60°$. Thus $DP = DM$ and $CB = BM$. Since the sum of the angles of a triangle equals a straight angle, we may proceed to evaluate the various angles as follows: $\angle BAC = 20°$, $\angle APD = \angle ABC = 80°$, $\angle EPM = 40°$, $\angle BEC = 50° = \angle BCE$. Hence, $BC = BE$, so $BE = BM$. Now in isosceles triangle MBE, $\angle MBE = 20°$, so $\angle BME = 80°$, whereupon $\angle EMP = 40°$. Consequently $EM = EP$, and DE is the perpendicular bisector of MP. Therefore, $\angle BMN = \frac{1}{2}(60°)$ or $30°$.



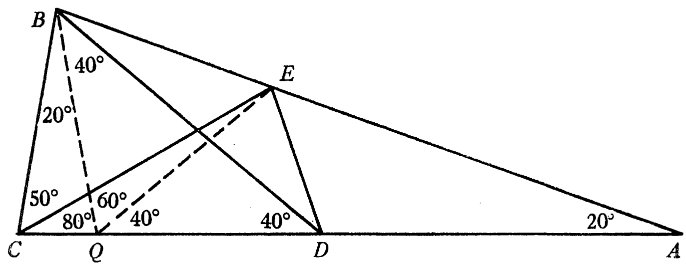
This method of solution will appear in my forthcoming book, Mathematical Quickies, McGraw-Hill, New York.

Method II. Consider a regular inscribed 18-gon $CBC_3C_4 \cdots C_{18}$ with center A. $\angle BCC_7 = \frac{1}{2}$ arc $BC_7 = 50°$. By symmetry, CC_7 and C_3C_{15} intersect at E. $\angle CBC_{13} = \frac{1}{2}$ arc $CC_{13} = 60°$. By symmetry BC_{13} and C_7C_{18} intersect at D, and $DB = DC_{18}$. Now $\angle DBA = 20° = \angle DAB$, so $DA = DB = DC_{18}$. Since $AC_{15}C_{18}C_3$ is a rhombus, C_3C_{15} perpendicularly bisects AC_{18}, so D falls on C_3C_{15}. Hence $\angle BDE = \frac{1}{2}$(arc BC_3 + arc $C_{13}C_{15}$) $= 30°$.



This method is adapted from the method of S. T. Thompson which appears without figure in Amer. Math. Monthly, 58 (January, 1951), 38.

Method III. Draw BQ intersecting CA at Q so that angle $CBQ = 20°$. Since the sum of the angles of a triangle equals a straight angle, we may proceed to evaluate the various angles as follows: $\angle CQB = 80°$, $\angle BCE = 50° = \angle BEC$. The triangles CBQ and CBE are isosceles, so $BQ = BC = BE$. Then triangle QBE is isosceles, and since $\angle QBE = 60°$, the triangle is equilateral with $BQ = BE = QE$. Again evaluating angles: $\angle BQE = 60°$, $\angle EQD = 40°$, $\angle QBD = 40° = \angle QDB$. Then triangle DQB is isosceles with $QD = QB$, and since $BQ = QE$, triangle DQE is isosceles. Consequently, $\angle QDE = \frac{1}{2}(180° - 40°)$ or 70°, and $\angle BDE = 30°$.



This method is adapted from that of Aaron Buchman, which appeared in School Science and Mathematics, 39 (April, 1939), 380.

The trigonometric solution of A. Sisk, requiring no construction lines, appears in Amer. Math. Monthly, 58 (January, 1951), 38.

Also solved by Leon Bankoff, Los Angeles, California (two solutions); Lindley J. Burton, Lake Forest College, Illinois; Mannis Charosh, New Utrecht High School, Brooklyn, New York; Michael Goldberg, Washington, D. C.; Bruce W. King, Ballston Lake, New York; J. D. E. Konhauser, University of Minnesota; Frank Morgan, Hanover, New Hampshire (two solutions); Stanley Rabinowitz, Far Rockaway, New York; Jerome J. Schneider, Chicago, Illinois; C. R. J. Singleton, Petersham, Surrey, England; Albert A. Schwartz, The City College, New York; Loretta G. Tabak, Carle Place, New York; James Vandergriff, University of Missouri at Rolla; Dale Woods, Missouri State Teachers College; and the proposer.

A number of solvers gave references to previous solutions in the 28th Yearbook of the National Council of Teachers of Mathematics, *Amer. Math. Monthly*, vol. 58, No. 1, January, 1951, and others.

COMMENT ON PROBLEM 84

84. [January, 1951, and September, 1965] *Proposed by Dewey Duncan, Los Angeles, California.*

We define a heterosquare as a square array of the first n^2 positive integers, so arranged that no two of the rows, columns, and diagonals (broken as well as straight) have the same sum.

(a) Show that no heterosquare of order two exists.
(b) Find a heterosquare of order three.

Comment by Prasert Na Nagara, Kasetsart University, Bangkok, Thailand.

Here are two "almost" heterosquares, i.e., arrays in which all sums but two are distinct.

127	983
485	625
693	417

These squares are "complements," i.e., each pair of numbers at the corresponding position adds up to 10. Let us use a collective term "a line" for a row, a column, a diagonal, or a broken diagonal. So, there are 12 lines, say $L_1, L_2, \cdots, L_{12}$ increasing in order of magnitude. If L_1 is $1+2+3=6$ the smallest L_2 and L_3 are $1+4+5=10$ and $2+4+6=12$ respectively. Similarly if $L_{12}=7+8+9=24$, the highest L_{11} and L_{10} must be 20 and 18 respectively. It is impossible to place 6 more distinct lines, L_4 to L_9 between L_3 and L_{10} having only 5 spaces between 12 and 18. We have only to investigate 19 triplets of L_1, L_2 and L_3 where $L_3 \leqq 11$. The case of $L_3 \geqq 12$ requires that $L_{10} \geqq 19$ in order to leave enough space for 6 lines (L_4 to L_9), and the complement of this square may be formed with $L_3 \leqq 30-19=11$. To help in finding all triplets whose $L_3 \leqq 11$, a table is formed.

Sum = 6	7	8	9	10	11
123	124	125	126	127	128
		134	135	136	137
			234	145	146
				235	236
					245

No two lines of a triplet may contain the same pair of numbers. Thus 124, 125 are not allowed, nor 124, 134. The only 19 triplets are given here.

124	124	125	125	125	134	134	134	134	134	126
135	235	234	234	234	126	126	127	127	235	145
236	137	136	137	146	235	245	236	245	128	137

126	135	135	135	234	234	234	234
235	127	127	127	127	136	145	145
137	146	236	245	146	128	128	137

Consider the first triplet 124, 135, 236. From the square

$$\begin{matrix} 1 & 2 & 4 \\ 3 & a & b \\ 5 & c & 6 \end{matrix}$$

six squares may be formed by substituting 7, 8, 9 for a, b, c in some orders. Placing of the triplet in different lines and orders merely gives the same 12 lines as those of the six squares. If all 3 lines of a triplet have a number in common, 8 squares must be checked. Thus only 118 squares must be investigated and we find that each has some equal lines. Thus, we may conclude that there are no heterosquares of order 3.

COMMENT ON Q375

Q375. Three nonzero numbers form a geometric progression whose ratio is an integer. If nine is added to the smallest number, an arithmetic progression is formed. Find all possible integral values for the numbers.

[*Submitted by Stanley Rabinowitz*]

Comment by Mark E. Kaminsky, Bronx High School of Science, New York.

The answer given has the solutions $(-1, -4, -16)$, $(-1, 2, -4)$, $(-9, -18, -36)$, $(2, -4, 8)$.

However, this only takes into consideration the case that the middle term in the geometric series is the middle term in the arithmetic series. However, this is not stated in the problem, and if we assume otherwise, four other solutions are obtained as follows.

Let a, ar, and ar^2 be the geometric series.

(1) Let a be the smallest, and let $a+9$ be the middle term of the arithmetic series.

Therefore: $ar + ar^2 = 2(a + 9)$

$$a(r^2 + r - 2) = 18$$

$$a(r - 1)(r + 2) = 18$$

$$r = 4, \quad a = 1, \quad ar = 4, \quad ar^2 = 16$$

$$r = -1, \quad a = -9, \quad ar = 9, \quad ar^2 = -9,$$

this gives us two solutions.

(2) Let a be the smallest term of the geometric series, and let ar^2 be the middle term of the arithmetic series.

Therefore:

$$ar + (a + 9) = 2ar^2$$

$$a(2r^2 - r - 1) = 9$$

$$a(2r + 1)(r - 1) = 9$$

$$r = -2, \quad a = 1, \quad ar = -2, \quad ar^2 = 4,$$

this gives us a third solution.

(3) Let ar^2 be the smallest term of the geometric series, and let ar^2+9 be the middle term of the arithmetic series.

Therefore:

$$a + ar = 2(ar^2 + 9)$$

$$-a(2r^2 - r - 1) = 18$$

$$a(2r + 1)(r - 1) = -18$$

$$r = -2, \quad a = -2, \quad ar = 4, \quad ar^2 = -8,$$

this is the fourth solution.

To summarize, the following solutions are not given in the answer of the author: (1, 4, 16), (−9, 9, −9), (1, −2, 4), and (−2, 4, −8). To the best of my knowledge, no other way of arranging the terms of the progressions yields a solution in integers.

QUICKIES

Q388. Evaluate in closed form the integral

$$\int_{\sqrt{2}}^{\infty} \frac{dx}{x + x^{\sqrt{2}}}$$

[*Submitted by Murray S. Klamkin*]

Q389. Compute

$$\lim_{\substack{x\to 0\\ y\to 0}}\left[\frac{x\cos y + y\sin y - xe^{-x}}{x^2+y^2}\right]$$

[*Submitted by Richard Sinkhorn*]

Q390. Let X be a topological space with c or fewer members [c the power of the continuum]. Prove that there is an open function from [0, 1] onto X.

[*Submitted by A. Wilansky*]

Q391. If a is odd and greater than four, $b=2^k$, k a natural number, then the arithmetic progression $a+bn$, $n=1, 2, 3, \cdots$, contains infinitely many primes by Dirichlet's theorem. Show, however, that at least one-third of the terms of the progression are composite in the sense that the set

$$S = \{a + bn \mid n = 1, 2, 3, \cdots, 3m\}$$

contains at least m composite numbers for each n.

[*Submitted by John D. Baum*]

Q392. Prove that $x^2-17y^2=11$ cannot be solved in integers.

[*Submitted by Mannis Charosh*]

(Answers on page 226)

ANNOUNCEMENT OF L. R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and this MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1966 recipients of these awards, selected by a committee consisting of R. P. Boas, Chairman; C. W. Curtis, and R. P. Dilworth, were announced by President Wilder at the Business Meeting of the Association on August 30, 1966, at Rutgers—The State University. The recipients of the Ford Awards for articles published in 1965 were the following:

C. B. Allendoerfer, Generalizations of Theorems About Triangles, this MAGAZINE, 38 (1965) 253–259.

P. D. Lax, Numerical Solution of Partial Differential Equations, MONTHLY, 72 (1965) Part II (Slaught Paper No. 10) 74–84.

Marvin Marcus and Henryk Minc, Permanents, MONTHLY, 72 (1965) 577–591.

HENRY L. ALDER, *Secretary*

ANSWERS

A388. Consider

$$I = \int \frac{dx}{x + x^m} = \int \frac{x^{-m}dx}{x^{1-m} + 1}$$
$$= \frac{1}{1 - m} \log (x^{1-m} + 1).$$

Thus

$$\int_{\sqrt{2}}^{\infty} \frac{dx}{x + x^{\sqrt{2}}} = (\sqrt{2} - 1) \log [1 + 2^{(1-\sqrt{2})/2}].$$

A389. Set $z = x + iy$ and let R denote the real part of z. Then the given limit may be written

$$\lim_{z \to 0} R\left[e^{-R(z)} \frac{e^z - 1}{z}\right]$$

which has the value 1.

A390. Let Q equal the rationals in $[0, 1]$. The natural map from $[0, 1]$ to $[0, 1]/Q$ is open. This latter space is indiscrete, therefore any map from it is open.

A391. It is sufficient to show that at least one of the integers S, $S+2^k$, $S+2^{k+1}$ is composite. If k is odd, then if $S \equiv 0 \pmod 3$, S is composite. If $S \equiv 1 \pmod 3$ then $S+2^k = 1-1 \equiv 0 \pmod 3$ and $S+2^{k+1}$ is composite. Finally if $S \equiv 2 \pmod 3$, $S+2^{k+1} \equiv 2+1 \equiv 0 \pmod 3$, and $S+2^{k+1}$ is composite. The argument is similar if k is even.

A392. For $x = 1, 2, \cdots, 8$, $x^2 \equiv 11 \pmod{17}$ is invalid. Therefore 11 is not a quadratic residue modulo 17.

DIAGONALIZING POSITIVE DEFINITE MATRICES

JAMES DUEMMEL, University of Montana

A diagonal matrix congruent to a given $n \times n$ symmetric matrix A may be constructed by the following procedure. With respect to the standard basis of n-tuple space, A represents a bilinear form $f(\alpha, \beta)$. Choose a new basis $\alpha_1, \cdots, \alpha_n$ where each α_j is in the null space of the linear functionals $f_i(\beta) = f(\alpha_i, \beta)$ for all i where $1 \leq i < j$. With respect to this new basis, $f(\alpha, \beta)$ will be represented by a diagonal matrix ([1], pp. 131 to 136).

If we regard vectors as $1 \times n$ matrices and denote the transpose of a matrix by a prime, then $f(\alpha, \beta)$ is the single entry in the 1×1 matrix $\alpha A \beta'$. In particular,

Q389. Compute

$$\lim_{\substack{x\to 0\\ y\to 0}}\left[\frac{x\cos y+y\sin y-xe^{-x}}{x^2+y^2}\right]$$

[*Submitted by Richard Sinkhorn*]

Q390. Let X be a topological space with c or fewer members [c the power of the continuum]. Prove that there is an open function from [0, 1] onto X.

[*Submitted by A. Wilansky*]

Q391. If a is odd and greater than four, $b=2^k$, k a natural number, then the arithmetic progression $a+bn$, $n=1, 2, 3, \cdots$, contains infinitely many primes by Dirichlet's theorem. Show, however, that at least one-third of the terms of the progression are composite in the sense that the set

$$S=\{a+bn \mid n=1, 2, 3, \cdots, 3m\}$$

contains at least m composite numbers for each n.

[*Submitted by John D. Baum*]

Q392. Prove that $x^2-17y^2=11$ cannot be solved in integers.

[*Submitted by Mannis Charosh*]

(Answers on page 226)

ANNOUNCEMENT OF L. R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and this MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1966 recipients of these awards, selected by a committee consisting of R. P. Boas, Chairman; C. W. Curtis, and R. P. Dilworth, were announced by President Wilder at the Business Meeting of the Association on August 30, 1966, at Rutgers—The State University. The recipients of the Ford Awards for articles published in 1965 were the following:

C. B. Allendoerfer, Generalizations of Theorems About Triangles, this MAGAZINE, 38 (1965) 253–259.

P. D. Lax, Numerical Solution of Partial Differential Equations, MONTHLY, 72 (1965) Part II (Slaught Paper No. 10) 74–84.

Marvin Marcus and Henryk Minc, Permanents, MONTHLY, 72 (1965) 577–591.

HENRY L. ALDER, *Secretary*